

# National Science and Mathematics Olympiad

Learning Materials for the Fourth Stage  
Finals of "NSMO"2026



Mathematics - Junior





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# Introduction

## **Our outstanding sons and daughters,**

We congratulate you on reaching the **pre-final stage** of the National Olympiad for Science and Mathematics, a stage that represents the culmination of your continuous efforts in understanding, training, and creativity.

In this special module, we will continue to deepen our exploration of the four branches:

**Combinatorics, Geometry, Algebra, and Number Theory**, with topics such as combinations, the Pythagorean theorem, quadratic equations, and common factors and multiples.

This stage aims to refine your **higher-level thinking skills** and **logical analysis**, and to train you to deal with complex problems that require precision and advanced reasoning.

It also serves as a direct preparation for the **final stage** of the competition, where true excellence is demonstrated through the ability to connect mathematical concepts and apply them in new situations.

We are confident that you are worthy of this stage, and we ask God to grant you success and guidance on your

**The Scientific Team for the National Science and Mathematics Olympiad (NSMO) – Mathematics Track**

## First Unit: ALGEBRA



## Squares of Binomials

If  $a$  and  $b$  are real numbers, then:

The square of a sum of two terms is given by the identity:

$$(a + b)^2 = a^2 + 2ab + b^2$$

The square of a difference of two terms is given by the identity:

$$(a - b)^2 = a^2 - 2ab + b^2$$

### Example 1:

**When Majed wants to square any number that ends with the digit 5, he uses the following steps:**

1. Erase the 5 at the end of the number to obtain a number  $k$ .
2. Multiply  $k$  by  $k + 1$  and write the digits 25 at the end of the resulting product.

For example, we find that  $65^2$  by evaluating  $6 \times 7 = 42$  and then placing 25 on the end of this product.

$$65^2 = 4225$$

How can that be correct?

### Solution:

$$\begin{aligned} 65^2 &= (60 + 5)^2 = 60^2 + 10 \times 60 + 25 = 60(60 + 10) + 25 \\ &= 60 \times 70 + 25 \\ &= 4200 + 25 = 4225 \end{aligned}$$

And in the same way,

$$\begin{aligned} 65^2 &= (70 - 5)^2 = 70^2 - 10 \times 70 + 25 = 70(70 - 10) + 25 \\ &= 70 \times 60 + 25 \\ &= 4200 + 25 = 4225 \end{aligned}$$

Use this trick to compute  $105^2$ .

## Exercises:

(1) In each of the following, complete the blank so that the expression becomes a perfect square:

(a)  $x^2 - 6x + \dots$

(b)  $x^2 + 7x + \dots$

(c)  $x^2 - 0.4x + \dots$

(d)  $x^2 - \dots + 42.25$

(2) Expand each of the following:

(a)  $(y + 5)^2$

(b)  $(3z + 8)^2$

(c)  $(x - 6)^2$

(d)  $(-2y + 9)^2$

(e)  $(-x - 9y)^2$

(f)  $(2r - \frac{2}{r})^2$

(3) Find  $l$  if

$$5l^2 - 20l = 0$$

(4) Find  $l$  if

$$l^2 - 144 = 0$$

(5) Find  $w$  if

$$29 = (w - 2)^2 - 7$$

(6) Find  $v$  if

$$94 - 5(v - 3)^2 = 14$$

(7) Find  $e$  if

$$3(4 + e)^2 - 40 = 68$$

(8) Find  $m$  if

$$m^2 - 6m + 9 = 0$$

(9) Find  $a$  if

$$a^2 + 36 = 12a$$

(10) Find  $t$  if

$$t^2 + 8t - 20 = 0$$

(11) Find  $h$  if

$$\frac{3(h-3)}{2} = \frac{27}{2h-6}$$

(12) Find  $x$  if

$$\frac{3x-6}{2} = \frac{27}{8x-16}$$

(13) Find  $y$  if

$$y^2 + 12y + 32 = 0$$

(14) If  $x - y = 8$ ,  $xy = -15$  find

a)  $x^2 + y^2$

b)  $(x + y)^2$

c)  $x^4 + y^4$

(15) If

$$x = 2025^{1447} - 2025^{-1} \quad \text{and} \quad y = 2025^{1447} + 2025^{-1447}$$

Find  $x^2 - y^2$ .

## Quadratic Equation

The Expression:  $ax^2 + bx + c = 0$

(Where  $a, b, c$  are constants and  $a \neq 0$ ) is called a **quadratic equation**. Its solutions are called **roots** or **zeros** of the polynomial.

There are several methods to find the roots of the equation  $ax^2 + bx + c = 0$  , including:

1. Factoring.
2. Completing the square.
3. The quadratic formula.

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

where the discriminant  $\Delta$  is given by:  $\Delta = b^2 - 4ac$

- If  $\Delta > 0$ : the equation has **two distinct real roots**.
- If  $\Delta < 0$ : the equation has **no real roots**.
- If  $\Delta = 0$ : the equation has **two equal real roots**.

### Problem 1:

Solve the equation:  $x^2 + 6x + 5 = 0$   
in three different ways.

**Solution:**

#### 1- Factoring.

$$x^2 + 6x + 5 = 0$$

We look for two numbers whose product is 5 and whose sum is 6.

$$\begin{aligned} \Rightarrow x^2 + 6x + 5 &= (x + 1)(x + 5) = 0 \\ \Rightarrow (x + 1) &= 0 \text{ or } (x + 5) = 0 \\ \Rightarrow x &= -1 \text{ or } x = -5 \end{aligned}$$

#### 2- Completing the square.

$$x^2 + 6x + 5 = 0$$

Move the constant term to the right side and add the square of half the coefficient of  $x$  to both sides.

$$\begin{aligned} \Rightarrow x^2 + 6x + 3^2 &= -5 + 3^2 \\ \Rightarrow (x + 3)^2 &= 4 \end{aligned}$$

Taking the square root of both sides:

$$\begin{aligned} \Rightarrow x + 3 &= \pm 2 \\ \Rightarrow x &= -1 \text{ or } x = -5 \end{aligned}$$

#### 3- The quadratic formula.

$$\begin{aligned} x^2 + 6x + 5 &= 0 \\ a = 1, b = 6, c = 5 &\Rightarrow \Delta = b^2 - 4ac = 6^2 - 4(1)(5) = 36 - 20 = 16 \\ x &= \frac{-b \pm \sqrt{\Delta}}{2a} \\ \Rightarrow x &= \frac{-6 \pm \sqrt{16}}{2(1)} \\ \Rightarrow x &= \frac{-6 + 4}{2} \text{ or } x = \frac{-6 - 4}{2} \\ \Rightarrow x &= -1 \text{ or } x = -5 \end{aligned}$$

#### Relationship Between Roots and Coefficients of a Quadratic Equation:

This is known as "**Vieta's Formulas**". If  $r, s$  are the roots of the quadratic equation:

$$ax^2 + bx + c = 0$$

Then the **sum of the roots** is given by the relation:  $r + s = \frac{-b}{a}$

And the **product of the roots** is given by the relation:  $rs = \frac{c}{a}$

### Exercises:

(1) Find all solutions to each of the following equations:

(a)  $x^2 - 12x - 540 = 0$

(b)  $3x^2 = 10x + 24$

(c)  $(x^4 - 11x^3 + 24x^2) - (4x^2 - 44x + 96) = 0$

(2) Find the value of  $a$  such that the equation:

$$ax^2 - 5x + 9 = 0$$

has exactly one real root.

(3) How many integers  $x$  satisfy the equation:

$$(x^2 - x - 1)^{x+2} = 1$$

(4) If one of the roots of the equation:

$$a(b - c)x^2 + b(c - a)x + c(a - b) = 0 \text{ is } x = 1.$$

Find the other root in terms of  $a, b$  and  $c$ .

(5) if  $a$  and  $b$  are the roots of the quadratic equation  $x^2 - mx + 2 = 0$ .

and given that  $a + \frac{1}{b}, b + \frac{1}{a}$  are the roots of the quadratic equation

$x^2 - px + q = 0$  find the value of  $q$ .

(6) Find all solutions for the following system of equations:

$$\begin{cases} x^2 + xy = 39 \\ x - y = -33 \\ y + z = 13 \end{cases}$$

(7) if  $p(x)$  is a second-degree polynomial such that

$$p(0) = -1, p(1) = 9, p(2) = 25$$

Find  $p(-1)$ .

(8) Find all values of  $k$  such that the equation:

$$x^2 + kx + 27 = 0$$

has two distinct real roots.

(9) Prove that if:

$$\frac{a+b}{a} = \frac{b}{a+b}$$

then it is impossible for both  $a$  and  $b$  to be real numbers.

(10) Find the real solutions for the equation:

$$(2 + (2 + (2 + (2 + x)^2)^2)^2)^2 = 15129$$

Given that:

$$(15129 = 123^2)$$

(11) Find all real solutions  $(x, y)$  that satisfy the system:

$$x^2 + y = 12 = y^2 + x$$

(12) Solve the following system of equations in the set of real numbers:

$$\begin{cases} 2x_1 = x_5^2 - 23 \\ 4x_2 = x_1^2 + 7 \\ 6x_3 = x_2^2 + 14 \\ 8x_4 = x_3^2 + 23 \\ 10x_5 = x_4^2 + 34 \end{cases}$$

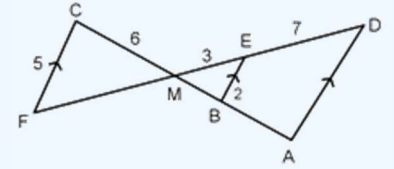
## Second Unit: Geometry



## Revision Exercises

(1) In the adjacent figure,  $AD \parallel BE \parallel FC$ . Use the given lengths on the figure to find the lengths

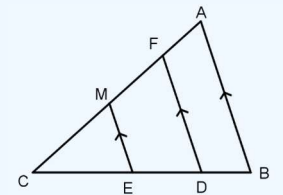
$$\overline{AB}, \overline{AD}, \overline{MB}, \overline{MF}$$



(2) In the adjacent figure,  $AB \parallel FD \parallel ME$ . Moreover,  $AF:FM:MC = 2:3:5$ .

If we have that  $ED = 7.5$  and  $AF = 4$ .

Find the length of  $\overline{EC}, \overline{BD}, \overline{AC}$



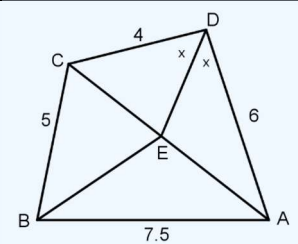
(3) Triangle  $ABC$  has side Lengths  $\overline{AB}, \overline{BC}, \overline{CA}$  equal to 4,5,6 respectively. The angle bisector of  $\angle A$  intersects  $\overline{BC}$  at  $D$ . Find the lengths of  $\overline{BD}, \overline{DC}$ .

(4) Triangle  $ABC$  has side Lengths  $\overline{AB}, \overline{BC}, \overline{CA}$  equal to 9,5,6 respectively. The external angle bisector of  $\angle A$  intersects  $\overline{BC}$  at  $D$ . Find the lengths of  $\overline{BD}, \overline{DC}$ .

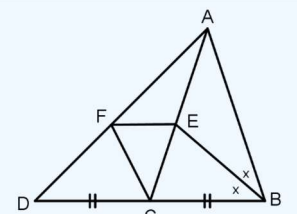
(5) In triangle  $ABC$ ,  $X$  is the midpoint of  $\overline{BC}$ . The angle bisector of  $\angle AXB$  intersects  $\overline{AB}$  at  $D$ . Similarly, the angle bisector of  $\angle AXC$  intersects  $\overline{AC}$  at  $E$ . Prove that  $\overline{DE} \parallel \overline{BC}$ .

(6) In the adjacent figure, quadrilateral  $ABCD$  has  $\overline{DE}$  which bisects  $\angle ADC$ .

Using the given lengths on the figure. Prove that  $\overline{BE}$  bisects  $\angle ABC$ .

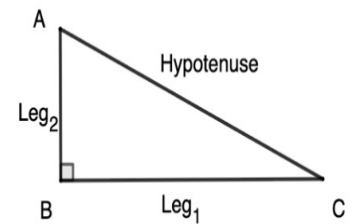


(7) In the adjacent figure, triangle  $ABC$  has  $AB = AC$ ,  $D$  lies on the ray  $\overline{BC}$  such that  $BC = CD$ . If  $\overline{BE}$  bisects  $\angle ABC$ , and  $EF \parallel BD$ . Prove that  $\overline{CF}$  bisects  $\angle ACD$ .



## Pythagorean Theorem and Its Applications

A right-angled triangle is a triangle in which one of its angles equals  $90^\circ$  (a right angle). The side opposite the  $90^\circ$  angle is called the hypotenuse, and the other two sides are called the legs (the sides that form the right angle).



The Pythagorean theorem defines the relationship between these three sides in any right-angled triangle.

### Theorem 1: (Pythagorean Theorem).

In a right-angle triangle, where  $a, b$  are the lengths of its legs and  $c$  is the length of its hypotenuse. The sum of the squares of the legs is equal to the square of the hypotenuse. That is:

$$a^2 + b^2 = c^2$$

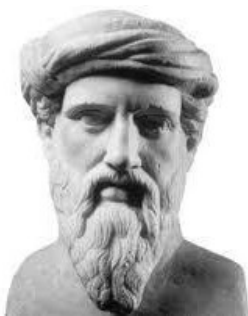
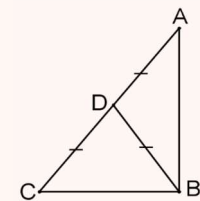
### Theorem 2: (Pythagorean Theorem Inverse).

Any triangle with side lengths  $a, b, c$  that satisfy the relation  $a^2 + b^2 = c^2$ . Then, this triangle is a right angles triangle. Moreover,  $c$  is the length of the hypotenuse.

### Theorem 3:

A triangle is right angled if one of the medians is equal to half of the side it is connected to. Moreover, that side will be the hypotenuse. For example:

$$(AD = BD = CD \Rightarrow \angle C = 90^\circ).$$



**\*Pythagoras** was a **Greek philosopher and mathematician** who lived approximately between **570 and 495 BC**. Born on the Greek island of **Samos**, he travelled extensively to many regions, including Greece, Egypt, and possibly India.

Around 530 BC, he settled in the Greek colony of **Croton, Italy**, where he established a school dedicated to the discussion of scientific and mathematical subjects. In his youth, he journeyed through **Mesopotamia** (modern-day Syria and Iraq) and spent time studying in **Egypt**. After two decades of travel and rigorous study, Pythagoras successfully absorbed all the mathematical knowledge known across the various major civilisations of the time.

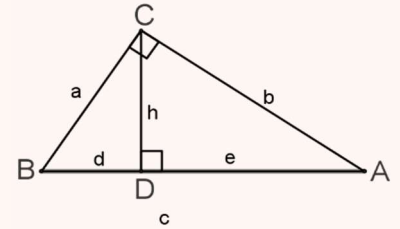
Pythagoras is credited with proving his most famous theorem in geometry—which will be a subject of our study in this program—by relating the areas of the squares corresponding to the sides of a **right-angled triangle**. This fundamental theorem continues to be used by many engineers today in the process of land and building construction.

**Theorem 4:**

In the right-angled triangle with sides  $a, b$  and hypotenuse  $c$ . If the altitude from the right angle to the hypotenuse

$DC = h$  and  $DA = e, DB = d$ . Then:

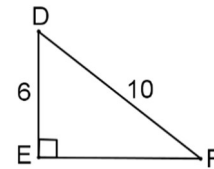
$$a^2 = dc, \quad b^2 = ec, \quad h^2 = ed, \quad hc = ab$$



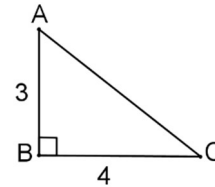
**Examples:**

Find the length of the unknown sides in the following triangles:

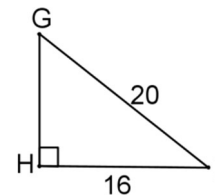
$$EF = \sqrt{DF^2 - DE^2} = \sqrt{100 - 36} = 8$$



$$AC = \sqrt{AB^2 + BC^2} = \sqrt{9 + 16} = 5$$



$$GH = \sqrt{GI^2 - HI^2} = \sqrt{144} = 12$$



### Exercises:

In (1 – 7). Find the value of  $x, y, z$ .

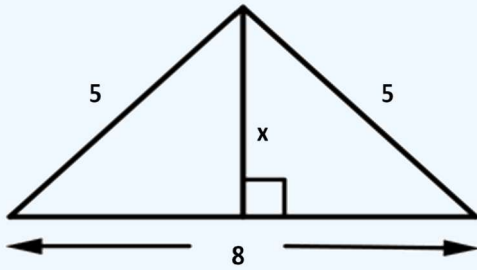
<p>(1)</p>	<p>(2)</p>
<p>(3)</p>	<p>(4)</p>
<p>(5)</p>	<p>(6)</p>
<p>(7)</p>	

In (8 – 11). You are given the length of a square's side. Find the length of its diagonal.

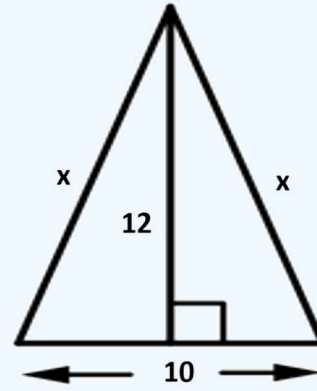
(8)	2	(9)	10	(10)	20k	(11)	$7n\sqrt{2}$
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In (12 – 15), find the value of  $x$ .

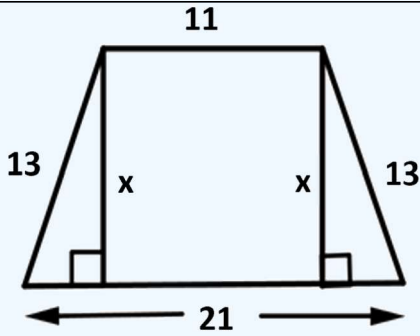
(12)



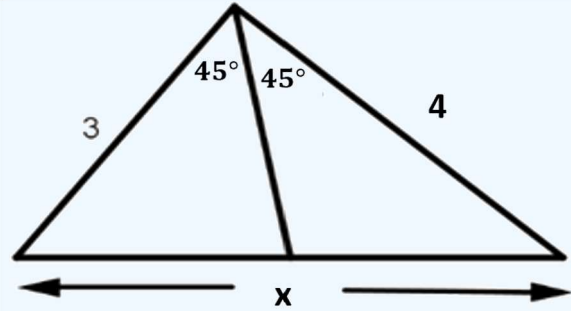
(13)



(14)



(15)



## Special Triangles

### Theorem 5:

In a right-angled triangle where the angles are equal to  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$ . The length of the side opposite the  $30^\circ$  angle is half the length of the hypotenuse. And the length of the side opposite to the  $60^\circ$  angle is  $\frac{\sqrt{3}}{2}$  the length of the hypotenuse.

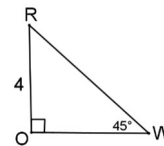
### Theorem 6:

In the isosceles right-angled triangle, the length of the sides opposite to the  $45^\circ$  angle is  $\frac{\sqrt{2}}{2}$  the length of the hypotenuse

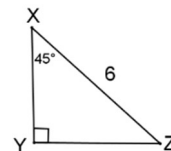
### Example:

Find the length of the unknown side in the following triangles:

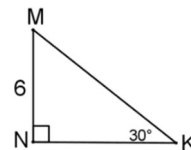
$$OW = 4, RW = \sqrt{16 + 16} = 4\sqrt{2}$$



$$YZ = YX = \frac{\sqrt{2}}{2} \cdot 6 = 3\sqrt{2}$$

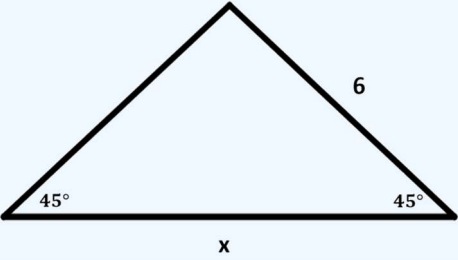
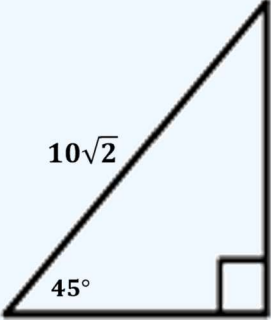
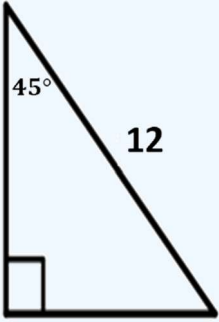
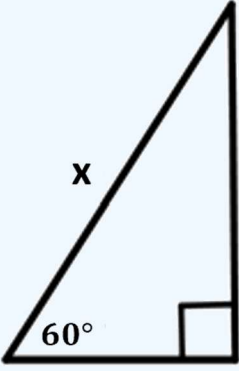
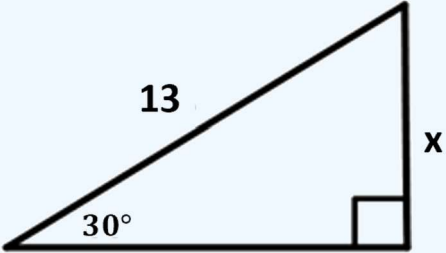
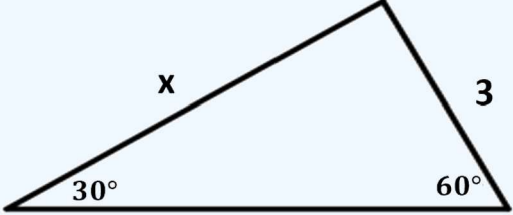
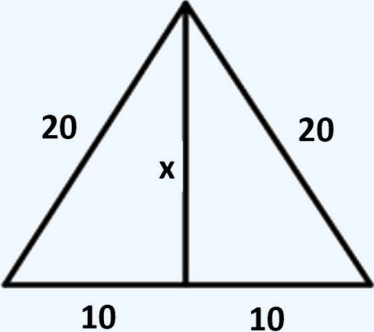


$$MK = 12, NK = \sqrt{144 - 36} = 6\sqrt{3}$$

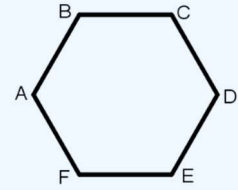


### Exercises:

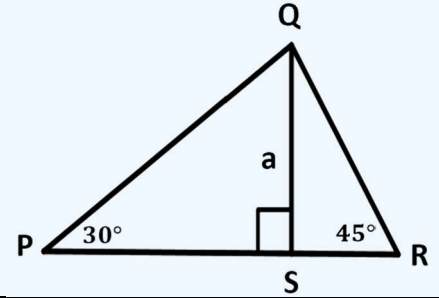
In exercises (1 – 7), find the value of  $x$ .

(1)		(2)	
(3)		(4)	
(5)		(6)	
(7)			

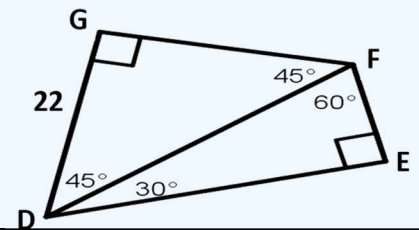
(8) In the adjacent figure:  $ABCDEF$  is a regular hexagon with side length 8. Find the length of  $AC, AD$



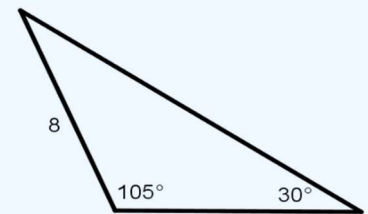
(9) In the following figure, find the lengths of  $PQ, PS, QR$ , in terms of  $a$ .



(10) In the adjacent figure, find the lengths of the unknown sides if possible.

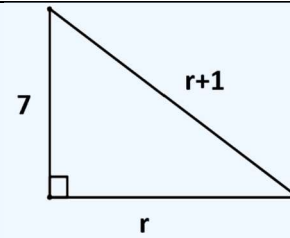


(11) In the adjacent figure, find the perimeter of the triangle.



## Challenge Exercises:

(1) In the adjacent figure, find  $r$ .



(2) The triangle  $ABC$  has  $\angle A = 90^\circ$ ,  $AB = AC$ . If  $D$  lies on  $BC$ .  
Prove that  $BD^2 + CD^2 = 2AD^2$ .

(3) If the perimeter of a right-angled triangle is  $30\text{ cm}$  and its area is  $30\text{ cm}^2$ .  
Find the lengths of its sides.

(4) The triangle  $ABC$  has  $\angle C = 90^\circ$ .  $AD$  is the internal angle bisector of  $\angle A$  intersects  $BC$  at  $D$ . If  
 $AB = 15$ ,  $AC = 9$ ,  $BD:DC = 5:3$   
Find the distance from  $D$  to  $AB$ .

(5) In triangle  $ABC$ , we have  $\angle C = 90^\circ$ ,  $BC = 6$ ,  $AC = 12$ .  
If the perpendicular bisector of  $AB$  intersects  $AB$ ,  $BC$  at  $D$ ,  $E$  respectively.  
Find the length of  $CE$ .

(6) In rectangle  $ABCD$ .  
We have  $CE \perp DB$  at  $E$ . If  $CE = 5$ ,  $BE = \frac{1}{4}BD$ ,  
find the length of  $AC$ .

(7) In triangle  $ABC$ , we have  $\angle C = 90^\circ$ . If point  $D$  is the midpoint of the segment  $AC$ .  
Prove that:

$$AB^2 + 3BC^2 = 4BD^2$$

(8) The triangle  $ABC$  has  $\angle C = 90^\circ$ . If  $E$ ,  $D$  lie on  $AC$ ,  $BC$  respectively.  
Prove that

$$AD^2 + BE^2 = AB^2 + DE^2$$

## Third Unit: Number Theory



## Revision Exercises

1) Which of the following numbers are prime numbers: 73,91,101,143,199?

2) Let  $p, q$  be different prime numbers. Find the number of different divisors for:

a)  $pq$     b)  $p^2q$     c)  $p^2q^2$     d)  $p^nq^m$

3) Prove that the multiplication of any three consecutive integers is divisible by 6.

4) Prove that the multiplication of any five consecutive integers is divisible by:

a) 30.

b) 120.

5) Find the least positive integer  $n$  such that 660 divides  $n!$ .

6) How many zeros are at the end of the number  $10!$ ?

7) Let  $n$  be a natural number. Is it possible for the decimal representation of  $(n!)$  to have exactly five consecutive zeros starting from the unit's digit??

8) Find all natural solutions  $x, y$  for the equation:

$$x^2 - y^2 = 33$$

## The Greatest Common Divisor ( $gcd$ )

### Definition:

Given  $a, b, c$  natural numbers such that  $a \times b = c$ . Then we say that  $a$  divides  $c$  and we denote that by  $a|c$ . Similarly,  $b|c$ . Moreover, we say that numbers  $a, b$  are divisors (or factors) of  $c$ . Finally, we also say that  $c$  is a multiple of  $a, b$ .

### Definition:

The greatest common divisor ( $gcd$  for short) of  $a, b$  is the largest positive integer that divides both  $a$  and  $b$

But how do we calculate the  $gcd$  of two numbers?

**Example 1:** Calculate  $gcd(6,8)$ .

**Solution:** We can look at the divisors of both 6 and 8:

$$d_6 = \{1,2,3,6\}$$

$$d_8 = \{1,2,4,8\}$$

Then, the common divisors for 6,8:

$$d_6 \cap d_8 = \{1,2\}$$

Thus, the greatest common divisor is 2

### Another way to calculate the greatest common divisor:

large numbers. Therefore, we use the prime factorization method, as it is more efficient.

#### This method is based on:

- Breaking each number down into its prime factors
- Identifying the common prime factors between the two numbers
- Choosing the **smallest exponent** for each common factor
- Multiplying these factors to obtain the greatest common divisor

The following example illustrates how to apply this method.

#### Example

2: Calculate  $gcd(36, 48)$ .

**Solution:** we factorise the two numbers into their prime factorisations.

To get that:

$$36 = 2^2 \times 3^2, \quad 48 = 2^4 \times 3^1$$

Now, the way to obtain the  $gcd$  prime factorization is by **taking the common prime factors of both numbers with the smaller power from each factor.**

Apply this rule in this example to get:

$$gcd(36, 48) = 2^2 \cdot 3^1 = 12$$

36	2	48	2
18	2	24	2
9	3	12	2
3	3	6	2
	1	3	3
			1

## Exercises:

1) Calculate  $gcd(8,9)$ .

2) Calculate  $gcd(54,96)$ .

3) Calculate  $gcd(35,91)$ .

4) Calculate  $gcd(6,54)$ .

5) Calculate  $gcd(199,256)$ .

6) The  $gcd$  of number  $n$  and 120 is 24. Which of the following could be the prime factorization of  $n$ ?

(a)  $2 \times 3^3$

(b)  $2^2 \times 3^3$

(c)  $2^3 \times 3^2 \times 11$

(d)  $2^4 \times 3^3 \times 5$

## The least common multiple (*lcm*)

### Definition:

The least common multiple (*lcm* for short) of  $a, b$  is the smallest positive integer that is a multiple of both  $a$  and  $b$ .

**Example 3:** Calculate  $lcm(6,8)$ .

**Solution:** We calculate the multiples of both 6 and 8 to get:

$$m_6 = \{6, 12, 18, 24, 30, 36, 42, 48, \dots\} \quad m_8 = \{8, 16, 24, 32, 40, 48, \dots\}$$

Then, we calculate the common multiples to obtain:  $m_6 \cap m_8 = \{24, 48, 72, \dots\}$

Therefore, the least common multiple for 6,8 is 24.

### Another way to calculate the least common multiple:

The multiples of any natural number form an infinite set, and listing many multiples to find the least common multiple can be difficult, especially for large numbers. Therefore, we use the **prime factorization method**, as it is more efficient.

### This method is based on:

- Breaking each number down into its prime factors
- Identifying **all** prime factors present in both numbers
- Choosing the **largest exponent** for each prime factor
- Multiplying these factors to obtain the least common multiple

**The following example illustrates how to apply this method.**

**Example 4:** Calculate  $lcm(36,48)$ .

**Solution:** we factorise the two numbers into their prime factorisations.

To get that:

$$36 = 2^2 \times 3^2, \quad 48 = 2^4 \times 3^1$$

Now, the way to obtain the  $lcm$  using the prime factorization is by **taking the common prime factors of both numbers with the higher power from each factor**. Apply this rule in this example to get:

$$lcm(36,48) = 2^4 \cdot 3^2 = 144$$

36	2	48	2
18	2	24	2
9	3	12	2
3	3	6	2
1		3	3
			1

## Exercises:

1) Calculate  $lcm$  (8,9).

2) Calculate  $lcm$  (54,96).

3) Calculate  $lcm$  (35,91).

4) Calculate  $lcm$  (6,54).

5) Calculate  $lcm$  (199,256).

6) The  $gcd$  of  $n$  and 20 is 180. Which of the following could be the prime factorisation of  $n$ ?

(a)  $2 \times 3^3$

(b)  $2^2 \times 3^2$

(c)  $2^3 \times 3^2$

(d)  $2^4 \times 3^3 \times 5$

## Important rule:

Given two natural numbers  $a, b$ , with  $d = \gcd(a, b)$  and  $m = \text{lcm}(a, b)$ . Then,

$$m \cdot d = a \cdot b$$

### Example 1:

If the  $\gcd(n, 8) = 2$ , and  $\text{lcm}(8, n) = 24$ . Then what is the value of  $n$ ?

### Solution:

Using the rule, we get that:

$$n \times 8 = 2 \times 24 \Rightarrow n = 6.$$

## Perfect Square:

Is a positive integer such that all the prime factors in its prime factorization have **even** power.

### For example:

The number  $A$  that has the prime factorisation:

$$A = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_n^{\alpha_n}$$

Is a perfect square if and only if  $\alpha_1, \alpha_2, \dots, \alpha_n$  were **even** numbers.

### Example 2:

Which of the following numbers are perfect squares:

$$144, 128, 1024, 360$$

### Solution:

By taking the prime factorization, we find that 144, 1024 are perfect squares while 128, 360 are not since they have primes with odd powers (show it on your own!).

## Exercises:

1) If the  $\gcd(n, 18) = 6$  and  $\text{lcm}(18, n) = 36$ . Then what is the value of  $n$ ?

2) Which of the following numbers are perfect squares:

196, 192, 169, 240

3) Challenge: Can a number that is composed of 100 zeros, 100 ones, 100 twos be a perfect square?

4) Challenge: Prove that the number with odd number of divisors must be a perfect square.

5) Challenge: Prove that for any natural numbers  $a, b$ , with  $d = \gcd(a, b)$  and  $m = \text{lcm}(a, b)$ . Then,

$$m \cdot d = a \cdot b.$$

## Fourth Unit: Combinatorics



## Combinations

**Combinations:** The number of ways to choose  $r$  items from a set of  $n$  items without regard to order and without repetition is  $\frac{n!}{(n-r)! \times r!}$

and this number is denoted by  ${}_n C_r$  or  $\binom{n}{r}$

(1) Five members of the Saudi team want to choose 2 of them at random to form a rules committee. In how many ways can they do that?

Properties of combinations and permutations

Compute the following and write your observations:

(a)  $(6)!$  and  $2! + 4!$       (b)  ${}_7 P_3$  and  $\frac{7!}{4!}$       (c)  ${}_8 P_3$  and  $3! \cdot \binom{8}{3}$       (d)  $\binom{8}{3}$  and  $\binom{8}{5}$

**Note:** From part (a) of the previous example we find that  $(2 + 4)! \neq 2! + 4!$

So we conclude that the statement  $(n + m)! \neq n! + m!$  is not always true.

And from parts (b), (c), and (d) we observe the equalities:

$${}_7 P_3 = \frac{7!}{4!} \quad ; \quad \text{and} \quad {}_8 P_3 = 3! \cdot \binom{8}{3} \quad \text{and} \quad \binom{8}{5} = \binom{8}{8-5} = \binom{8}{3}$$

We will find later that these relations are always true for any positive integers  $m, n$  such that  $m \leq n$

$$n! = n \cdot (n - 1)!$$

$${}_n P_m = m! \cdot \binom{n}{m}$$

$$\binom{n}{m} = \binom{n}{n - m}$$

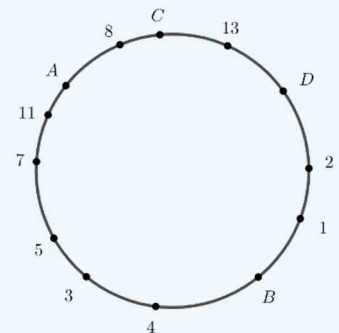
## Exercises:

(2) How many different words can be formed from rearranging the letters of *ELEMENTARY* such that no two *E*'s appear together?

(3) How many different 10-letter words can be formed using only the letters *E* and *F*, if the number of *E*'s is greater than the number of *F*'s?

(4) Points are labeled on the circumference of a circle as shown. We will draw chords according to these conditions:

1. Draw a chord between any two points whose names are letters.
2. Draw a chord between any two points whose names are odd numbers.
3. Draw a chord between any two points whose names are even numbers.



How many chords can be drawn in total?

(5) In how many ways can 8 students be divided into:

- (a) Two distinct groups of 3 students each, with 2 students left unassigned?  
(Note: In the next four cases, the groups are not distinct.)
- (b) Two groups of 4 students each?
- (c) Four groups of 2 students each?
- (d) One group of 4 students and two groups of 2 students?
- (e) One group of 4 students, one group of 3 students, and one student left unassigned?

(6) Osama wants to sit with 10 of his friends around a round table. How many different pairs of Osama's friends can be seated next to him?

## Exercises on Counting Principles

(1) We have three rooms: the first holds 1 person, the second holds 2 people, and the third holds 4 people. If we want to accommodate 7 students in these three rooms, in how many different ways can this be done?

(2) How many 10-digit positive integers contain at least two identical digits?

(3) How many strings can be formed using only the letters A and B, containing exactly five A's and at most three B's?

(4) Is the number of 7-digit positive integers that do not contain the digit 1 equal to half the total number of 7-digit positive integers?

(5) How many 10-digit positive integers have an even digit sum?

(6) In how many ways can we place 4 identical rooks (same color) on a chessboard so that no two rooks attack each other?

(7) We have 6 distinct books to arrange on a shelf, with the condition that two specific books (for example, A and B) must always be next to each other. How many arrangements are possible?

(8) How many ways are there to choose a 3-person committee from 10 people, given that Jawad must be on the committee?

## Ascending and Descending Order

In this lesson, we learn how to count numbers whose digits are arranged in **ascending** or **descending** order. We will see that this ordering affects the total number of possible outcomes. These ideas will be applied through examples and various exercises.

### Example 1:

In how many ways can a 4-digit number be formed using the digits  $\{1, 2, 3, 4, 5, 6\}$  such that its digits are arranged in ascending order?

#### Solution:

We have the digits:  $\{1, 2, 3, 4, 5, 6\}$

We want to form a **4-digit number** such that the digits are:

- chosen from this set
- arranged in ascending order

Important note: Ascending or descending order means that **no digit can be repeated**, since the digits must be distinct.

When the digits are **arranged in ascending order**, this means that:

- We choose 4 distinct digits out of the 6 available digits.
- **The order is then automatically determined** (from smallest to largest).

Therefore, the problem reduces to:

In how many ways can we choose 4 digits from 6 without regard to order?

This is exactly the definition of **combinations**.

$$\binom{6}{4} = \frac{6!}{4! 2!} = \frac{6 \times 5}{2 \times 1} = 15$$

#### What if the set contains zero digits?

What difference will appear in the solution?

We illustrate this with the following example:

### Example 2:

In how many ways can a 4-digit number be formed using the digits  $\{0, 1, 2, 3, 4, 5, 6\}$  such that its digits are arranged in **ascending order**?

#### Solution:

The set is  $\{0, 1, 2, 3, 4, 5, 6\}$

We want to form a **4-digit number** whose digits are arranged in **ascending order**.

#### Important note:

Saying "a **4-digit number**" means that the number must indeed have **four digits**, so:

- The first digit **cannot be zero**.

And since the digits are **arranged in ascending order**:

The first digit represents the smallest digit in the number.

Therefore, zero cannot be used, because if it were used, it would appear in the first position, making the number not a four-digit number. Hence, we exclude zero, and the set becomes:  $\{1, 2, 3, 4, 5, 6\}$

At that point:

- We choose only 4 digits out of the 6,
- and their ascending order is automatically determined.

Therefore, we use combinations, just as in the previous example.

$$\binom{6}{4}$$

#### Conclusion:

From the above, we conclude that when the digits are arranged in **ascending order**, the presence of zero affects the solution. This is because zero is the smallest digit and would occupy the first position, which prevents the formation of a number with the required number of digits. Hence, zero must be excluded when making the selection. On the other hand, when the digits are arranged in **descending order**, the method of solution remains the same, and the presence or absence of zero does not affect the result. This is because the first position is occupied by the largest digit, and zero can never appear there.

### Exercises:

(1) How many numbers from 100,101, ..., 999 have their three digits arranged in ascending or descending order from left to right?

(2) In how many ways can a 5-digit number be formed using digits from  $\{1, 2, 3, 4, 5, 6, 7, 8\}$  such that its digits are:

- (a) in ascending order?
- (b) in descending order?

(3) In how many ways can a 4-digit number be formed using digits from  $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$  such that its digits are:

- (a) in ascending order?
- (b) in descending order?

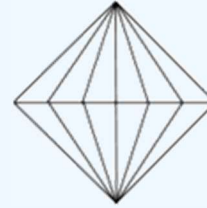
(4) Mohammed has ten cards numbered from 1 to 10. He plays a game with his brother Salem, where Salem draws three cards in a row without replacement. Salem wins if the three cards he draws are in ascending order; otherwise, Mohammed wins. Who has the higher chance of winning, Salem or Mohammed? And what is each player's winning probability (percentage)?



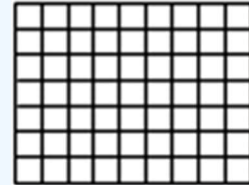
### Exercises:

(1) A frog starts at the origin 0 on the number line. Each time it jumps one step, either forward (+1) or backward (-1). After 13 jumps, in how many different ways can it end up at position -1?

(2) How many triangles are there in the following figure?



(3) How many rectangles are there in the following figure?



(4) In a certain city, a license plate consists (from left to right) of two letters in alphabetical order, followed by two digits in increasing order (for example, *RE64*). If the letter *O* is not allowed and the digit *0* is not used, how many different license plates are possible?

(5) On a refrigerator door, there are 9 magnetic tiles, each labeled with one of the letters from the word *MATHCOUNT*. We select two vowels and three consonants from these tiles and place them into a bag. Assuming that the two *T* letters are indistinguishable, how many different possible collections of letters can be in the bag?

# Solutions



## Algebra Solutions

### Squares of Binomials:

#### Exercises:

(1)

By using the identity:

$$x^2 \pm bx + \left(\frac{b}{2}\right)^2 = \left(x \pm \frac{b}{2}\right)^2$$

We get

(a)  $x^2 - 6x + 9$

(b)  $x^2 + 7x + \frac{49}{4}$

(c)  $x^2 - 0.4x + 0.04$

(d)  $x^2 - 13x + 42.25$

(2)

(a)  $(y + 5)^2 = y^2 + 10y + 25$

(b)  $(3z + 8)^2 = 9z^2 + 48z + 64$

(c)  $(x - 6)^2 = x^2 - 12x + 36$

(d)  $(-2y + 9)^2 = 4y^2 - 36y + 81$

(e)  $(-x - 9y)^2 = x^2 + 18xy + 81y^2$

(f)  $\left(2r - \frac{2}{r}\right)^2 = 4r^2 - 8 + \frac{4}{r^2}$

(3)

$$5l^2 - 20l = 0$$

$$\Rightarrow 5l(l - 4) = 0$$

$$\Rightarrow 5l = 0 \quad \text{or} \quad l - 4 = 0$$

$$\Rightarrow l = 0 \quad \text{or} \quad l = 4$$

(4)

$$l^2 - 144 = 0$$

$$\Rightarrow l^2 - 12^2 = 0$$

$$\Rightarrow (l - 12)(l + 12) = 0$$

$$\Rightarrow l - 12 = 0 \quad \text{or} \quad l + 12 = 0$$

$$\Rightarrow l = \pm 12$$

(5)

$$29 = (w - 2)^2 - 7$$

$$\Rightarrow (w - 2)^2 = 36$$

$$\Rightarrow w - 2 = 6 \quad \text{or} \quad w - 2 = -6$$

$$\Rightarrow w = 8 \quad \text{or} \quad w = -6$$

(6)

$$94 - 5(v - 3)^2 = 14$$

$$\Rightarrow 5(v - 3)^2 = 80$$

$$\Rightarrow (v - 3)^2 = 16$$

$$\Rightarrow v - 3 = 4 \quad \text{or} \quad v - 3 = -4$$

$$\Rightarrow v = 7 \quad \text{or} \quad v = -1$$

(7)

$$3(4 + e)^2 - 40 = 68$$

$$\Rightarrow 3(4 + e)^2 = 108$$

$$\Rightarrow (4 + e)^2 = 36$$

$$\Rightarrow 4 + e = 6 \quad \text{or} \quad 4 + e = -6$$

$$\Rightarrow e = 2 \quad \text{or} \quad e = -10$$

(8)

$$m^2 - 6m + 9 = 0$$

$$\Rightarrow (m - 3)^2 = 0$$

$$\Rightarrow m - 3 = 0$$

$$\Rightarrow m = 3$$

(9)

$$a^2 + 36 = 12a$$

$$\Rightarrow a^2 - 12a + 36 = 0$$

$$\Rightarrow (m - 6)^2 = 0$$

$$\Rightarrow m - 6 = 0$$

$$\Rightarrow m = 6$$

(10)

$$t^2 + 8t - 20 = 0$$

$$\Rightarrow (t - 2)(t + 10) = 0$$

$$\Rightarrow t - 2 = 0 \quad \text{or} \quad t + 10 = 0$$

$$\Rightarrow t = 2 \quad \text{or} \quad t = -10$$

(11)

$$\frac{3(h - 3)}{2} = \frac{27}{2h - 6}$$

$$\Rightarrow (h - 3)(h - 3) = \frac{2 \cdot 27}{2 \cdot 3}$$

$$\Rightarrow (h - 3)^2 = 9$$

$$\Rightarrow h - 3 = 3 \quad \text{or} \quad h - 3 = -3$$

$$\Rightarrow h = 6 \quad \text{or} \quad h = 0$$

(12)

$$\frac{3x - 6}{2} = \frac{27}{8x - 16}$$

$$\Rightarrow (x - 2)(x - 2) = \frac{2 \cdot 27}{8 \cdot 3}$$

$$\Rightarrow (x - 2)^2 = \frac{9}{4}$$

$$\Rightarrow x - 2 = \frac{3}{2} \quad \text{or} \quad x - 2 = -\frac{3}{2}$$

$$\Rightarrow x = \frac{7}{2} \quad \text{or} \quad x = \frac{1}{2}$$

(13)

$$y^2 + 12y + 32 = 0$$

$$\Rightarrow (y + 8)(y + 4) = 0$$

$$\Rightarrow y + 8 = 0 \quad \text{or} \quad y + 4 = 0$$

$$\Rightarrow y = -8 \quad \text{or} \quad y = -4$$

**(14)**

If  $x - y = 8$  and  $xy = -15$  find:

a)  $x^2 + y^2$

b)  $(x + y)^2$

c)  $x^4 + y^4$

a)  $(x - y)^2 = 8^2$

$$\Rightarrow x^2 - 2xy + y^2 = 64$$

$$\Rightarrow x^2 - 2(-15) + y^2 = 64$$

$$\Rightarrow x^2 + y^2 = 64 - 30 = 34$$

b)  $(x + y)^2 = x^2 + 2xy + y^2$

$$\Rightarrow (x + y)^2 = x^2 + 2(-15) + y^2$$

$$\Rightarrow (x + y)^2 = x^2 + y^2 - 30$$

$$\Rightarrow (x + y)^2 = 34 - 30 = 4$$

c)  $(x^2 + y^2)^2 = 34^2$

$$\Rightarrow x^4 + 2(xy)^2 + y^4 = 34^2$$

$$\Rightarrow x^4 + y^4 = 1156 - 450 = 706$$

**(15)**

$$x - y = 2025^{1447} - 2025^{-1447} - 2025^{1447} - 2025^{-1447} = -\frac{2}{2025^{1447}}$$

$$x + y = 2025^{1447} - 2025^{-1447} + 2025^{1447} + 2025^{-1447} = 2 \cdot 2025^{1447}$$

$$\therefore x^2 - y^2 = (x - y)(x + y)$$

$$\therefore x^2 - y^2 = \left(-\frac{2}{2025^{1447}}\right)(2 \cdot 2025^{1447}) = -4$$

## Quadratic Equation:

### Exercises:

(1)

$$(a) x^2 - 12x - 540 = 0$$

We are looking for two numbers whose product is  $-540$  and whose sum is  $-12$

The two numbers are  $18$  and  $-30$

$$\Rightarrow x^2 - 12x - 540 = (x - 30)(x + 18) = 0$$

$$\Rightarrow x = 30 \text{ or } x = -18$$

$$(b) 3x^2 = 10x + 24$$

$$\Rightarrow 3x^2 - 10x - 24 = 0$$

"Move all terms to one side of the equation"

$$a = 3, b = -10, c = -24 \Rightarrow \Delta = b^2 - 4ac = (-10)^2 - 4(3)(-24) = 100 + 288 = 388$$

$$\therefore x = \frac{10 \pm \sqrt{388}}{6} = \frac{5 \pm \sqrt{97}}{3}$$

$$\Rightarrow x = \frac{5 + \sqrt{97}}{3} \text{ or } x = \frac{5 - \sqrt{97}}{3}$$

$$(c) (x^4 - 11x^3 + 24x^2) - (4x^2 - 44x + 96) = 0$$

$$\Rightarrow x^2(x^2 - 11x + 24) - 4(x^2 - 11x + 24) = 0$$

$$\Rightarrow (x^2 - 11x + 24)(x^2 - 4) = 0$$

$$\Rightarrow (x - 8)(x - 3)(x - 2)(x + 2) = 0$$

$$\Rightarrow x = 8 \text{ or } 3 \text{ or } 2 \text{ or } -2$$

(2)

$$ax^2 - 5x + 9 = 0$$

The equation has exactly one solution if the discriminant equals zero ( $\Delta = 0$ ) that is:

$$\Delta = b^2 - 4ac = (-5)^2 - 4(a)(9) = 0$$

$$\Rightarrow 25 - 36a = 0$$

$$\Rightarrow a = \frac{25}{36}$$

**(3)**

$$(x^2 - x - 1)^{x+2} = 1$$

**Case 1:** The exponent is equal to zero (provided the base is non-zero)

$$x + 2 = 0 \Rightarrow x = -2$$

When  $x = -2$  the base is non-zero.

**Case 2:** The base is equal to 1.

$$x^2 - x - 1 = 1 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2 \text{ or } x = -1$$

**Case 3:** The base is equal to  $(-1)$  (provided the exponent is an even number)

$$x^2 - x - 1 = -1 \Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0 \text{ or } x = 1$$

**When  $x = 0$**  the exponent is  $x + 2 = 0 + 2 = 2$  (an even number).

Therefore, the equation holds.

**When  $x = 1$**  the exponent is  $x + 2 = 1 + 2 = 3$  (an odd number).

Therefore, the equation does not hold.

**Conclusion:** The integer values of  $x$  that satisfy the equation are  $\{-2, 2, -1, 0\}$  totaling four values.

**(4)**

Using Vieta's Formulas. Let the second root of the equation:

$$a(b - c)x^2 + b(c - a)x + c(a - b) = 0$$

be  $r$  The product of the roots is given by the relation:

$$r \cdot 1 = r = \frac{c(a - b)}{a(b - c)}$$

(5)

Since  $a$  and  $b$  are the roots of  $x^2 - mx + 2 = 0$ , the product of the roots is:

$$ab = 2$$

Given that  $a + \frac{1}{b}$  and  $b + \frac{1}{a}$  are roots of  $x^2 - px + q = 0$ , their product is  $q$ :

$$\begin{aligned} \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) &= ab + 2 + \frac{1}{ab} = q \\ \Rightarrow q &= 2 + 2 + \frac{1}{2} = \frac{9}{2} \end{aligned}$$

(6)

$$\begin{cases} x^2 + xy = 39 \longrightarrow (1) \\ x - y = -33 \longrightarrow (2) \\ y + z = 13 \longrightarrow (3) \end{cases}$$

From equation (2), we find that  $y = x + 33$ . Substituting this into equation(1), we obtain:

$$\begin{aligned} x^2 + x(x + 33) &= 39 \\ \Rightarrow x^2 + x^2 + 33x - 39 &= 0 \\ \Rightarrow 2x^2 + 33x - 39 &= 0 \end{aligned}$$

Using the **Quadratic Formula**, where  $a = 2, b = 33, c = -39$

$$\Delta = b^2 - 4ac = 33^2 - 4(2)(-39) = 1401$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{\Delta}}{2a} \\ \Rightarrow x &= \frac{-33 \pm \sqrt{1401}}{2(2)} \\ \Rightarrow x &= \frac{-33 \pm \sqrt{1401}}{4} \end{aligned}$$

By substituting the values of  $x$  back into equations (2) and (3), we find  $y$  and  $z$ :

$$\begin{aligned} y = x + 33 &\Rightarrow y = \frac{-33 \pm \sqrt{1401}}{4} + 33 = \frac{99 \pm \sqrt{1401}}{4} \\ z = 13 - y &\Rightarrow z = 13 - \frac{99 \pm \sqrt{1401}}{4} = \frac{-47 \mp \sqrt{1401}}{4} \end{aligned}$$

Therefore, the system has two solutions:

$$\left(\frac{-33 + \sqrt{1401}}{4}, \frac{99 + \sqrt{1401}}{4}, \frac{-47 - \sqrt{1401}}{4}\right), \left(\frac{-33 - \sqrt{1401}}{4}, \frac{99 - \sqrt{1401}}{4}, \frac{-47 + \sqrt{1401}}{4}\right)$$

(7)

Assume the equation is of the form  $p(x) = ax^2 + bx + c = 0$ . Since  $P(0) = -1$ , then:

$$P(0) = a(0)^2 + b(0) + c = -1 \Rightarrow c = -1$$

Given that  $P(1) = 9$ , then:

$$P(1) = a(1)^2 + b(1) - 1 = 9 \Rightarrow a + b = 10 \longrightarrow (1)$$

Given that  $P(2) = 25$ , then:

$$P(2) = a(2)^2 + b(2) - 1 = 25 \Rightarrow 4a + 2b = 26 \Rightarrow 2a + b = 13 \longrightarrow (2)$$

By subtracting equation (1) from equation (2), we obtain:

$$a = 3 \Rightarrow b = 7$$

Therefore:

$$P(-1) = 3(-1)^2 + 7(-1) - 1 = 3 - 7 - 1 = -5$$

(8)

For the equation  $x^2 + kx + 27 = 0$  to have two distinct real roots, the discriminant  $\Delta > 0$

$$\Delta = k^2 - 4(1)(27) = k^2 - 108 > 0$$

$$\Rightarrow k^2 > 108 \Rightarrow |k| > \sqrt{108} = 6\sqrt{3}$$

Hence:

$$k > 6\sqrt{3} \text{ or } k < -6\sqrt{3}$$

(9)

$$\frac{a+b}{a} = \frac{b}{a+b}$$

$$\Rightarrow (a+b)^2 = ab$$

$$\Rightarrow a^2 + 2ab + b^2 = ab$$

$$\Rightarrow a^2 + ab + b^2 = 0$$

Using the **Quadratic Formula** to solve for  $a$ , we find the discriminant:

$$\Delta = (b)^2 - 4(1)(b^2) = -3b^2$$

- If  $b \neq 0$  then  $\Delta < 0$ . Consequently, there are no real solutions for  $a$ .
- If  $b = 0$ , the original equation becomes  $\frac{a}{a} = \frac{0}{a} \Rightarrow 1 = 0$  This is impossible.

Therefore,  $a$  and  $b$  cannot both be real numbers.

**(10)**

$$(2 + (2 + (2 + (2 + x)^2)^2)^2)^2 = 15129$$

By taking the square root of both sides (neglecting the negative result since the left-hand side is positive):

$$\Rightarrow 2 + (2 + (2 + (2 + x)^2)^2)^2 = 123$$

$$\Rightarrow (2 + (2 + (2 + x)^2)^2)^2 = 121$$

Taking the square root again:

$$\Rightarrow 2 + (2 + (2 + x)^2)^2 = 11$$

$$\Rightarrow (2 + (2 + x)^2)^2 = 9$$

Following the same procedure:

$$\Rightarrow 2 + (2 + x)^2 = 3$$

$$\Rightarrow (2 + x)^2 = 1$$

$$\Rightarrow 2 + x = \pm 1$$

$$\Rightarrow x = -1 \text{ or } -3$$

**(11)**

Given:  $x^2 + y = 12 = y^2 + x$  Thus,  $x^2 + y = y^2 + x$

$$\Rightarrow x^2 - y^2 + y - x = 0$$

$$\Rightarrow (x^2 - y^2) - (x - y) = 0$$

$$\Rightarrow (x - y)(x + y) - (x - y) = 0$$

$$\Rightarrow (x - y)(x + y - 1) = 0$$

This gives us two cases:

**1)  $x - y = 0$**

$$\Rightarrow x = y$$

Substituting into the first equation:

$$x^2 + x = 12 \Rightarrow x^2 + x - 12 = 0 \Rightarrow (x + 4)(x - 3) = 0 \Rightarrow (x, y) = (-4, -4), (3, 3)$$

**2)  $x + y - 1 = 0$**

$$\Rightarrow y = 1 - x$$

Substituting into the first equation:

$$x^2 + 1 - x = 12 \Rightarrow x^2 - x - 11 = 0$$

Using the quadratic formula where  $a = 1, b = -1, c = -11$

$$\Delta = b^2 - 4ac = 1 - 4(1)(-11) = 45$$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$\Rightarrow x = \frac{1 \pm 3\sqrt{5}}{2}$$

$$\Rightarrow y = 1 - \frac{1 \pm 3\sqrt{5}}{2} = \frac{1 \mp 3\sqrt{5}}{2}$$

The solution set for the system is:

$$(x, y) = (-4, -4), (3, 3), \left( \frac{1 + 3\sqrt{5}}{2}, \frac{1 - 3\sqrt{5}}{2} \right), \left( \frac{1 - 3\sqrt{5}}{2}, \frac{1 + 3\sqrt{5}}{2} \right)$$

(12)

$$\begin{cases} 2x_1 = x_5^2 - 23 \\ 4x_2 = x_1^2 + 7 \\ 6x_3 = x_2^2 + 14 \\ 8x_4 = x_3^2 + 23 \\ 10x_5 = x_4^2 + 34 \end{cases}$$

Summing all equations and rearranging terms:

$$(x_1^2 - 2x_1) + (x_2^2 - 4x_2) + (x_3^2 - 6x_3) + (x_4^2 - 8x_4) + (x_5^2 - 10x_5) + 55 = 0$$

By completing the square for each expression in parentheses:

$$(x_1^2 - 2x_1 + 1) + (x_2^2 - 4x_2 + 4) + (x_3^2 - 6x_3 + 9) + (x_4^2 - 8x_4 + 16) + (x_5^2 - 10x_5 + 25) = 0$$

$$\Rightarrow (x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2 + (x_4 - 4)^2 + (x_5 - 5)^2 = 0$$

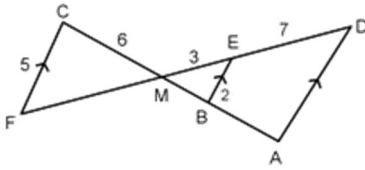
Since the sum of squares equals zero, each individual term must be zero:

$$x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$$

## Geometry Solutions

### Revision Exercises Solutions

(1)



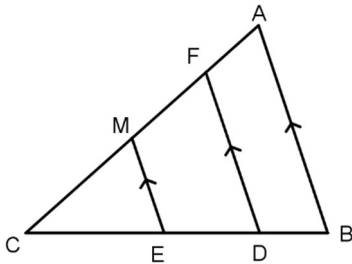
$$\frac{ME}{MF} = \frac{EB}{FC} \Rightarrow MF = \frac{5 \times 3}{2} = 7.5$$

$$\frac{MB}{MC} = \frac{EB}{FC} \Rightarrow MB = \frac{6 \times 2}{5} = 2.4$$

$$\frac{ME}{ED} = \frac{MB}{BA} \Rightarrow BA = \frac{7 \times 2.4}{3} = 5.6$$

$$\frac{BE}{AD} = \frac{ME}{MD} \Rightarrow AD = \frac{2 \times 10}{3} = \frac{20}{3}$$

(2)

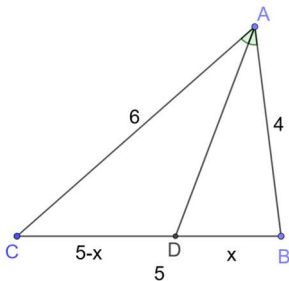


$$\frac{MC}{EC} = \frac{MF}{ED} \Rightarrow CE = \frac{5 \times 7.5}{3} = 12.5$$

$$\frac{AF}{DB} = \frac{MF}{ED} \Rightarrow DB = \frac{2 \times 7.5}{3} = 5$$

$$\frac{AF}{AC} = \frac{2}{10} \Rightarrow AC = \frac{4 \times 10}{2} = 20$$

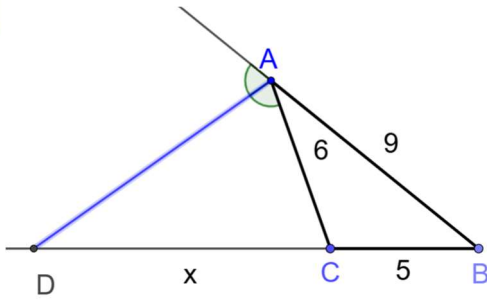
(3)



$$\frac{4}{6} = \frac{BD}{DC} \Rightarrow \frac{4}{6} = \frac{x}{5-x}$$

$$\Rightarrow BD = x = 2$$

(4)

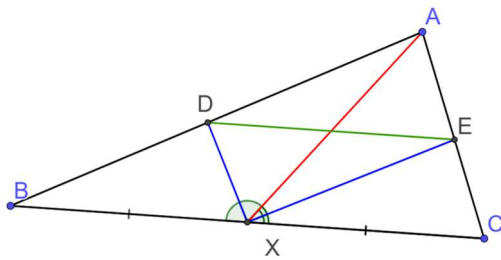


$$\frac{AB}{AC} = \frac{DB}{DC} \Rightarrow \frac{9}{6} = \frac{5+x}{x}$$

$$\Rightarrow 9x = 30 + 6x \Rightarrow x = 10$$

$$\Rightarrow DC = 10, BD = 15$$

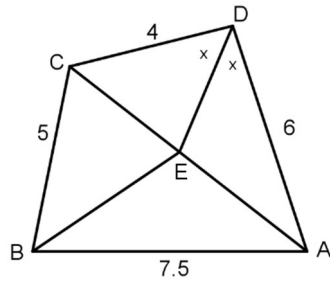
(5)



$$\frac{AX}{XC} = \frac{AE}{EC} \quad (1) \quad \frac{AX}{XB} = \frac{AD}{DB} \quad (2)$$

$$XB = XC \Rightarrow \frac{AE}{EC} = \frac{AD}{DB} \Rightarrow ED \parallel BC$$

(6)

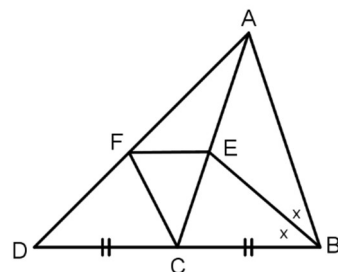


$$\frac{AE}{EC} = \frac{AD}{DC} = \frac{6}{4} = \frac{3}{2} \quad (1) \quad \frac{AB}{BC} = \frac{7.5}{5}$$

$$= \frac{3}{2} \quad (2)(1), (2) \Rightarrow \frac{AB}{BC} = \frac{AE}{EC}$$

From the inverse of the angle bisector theorem we have  $BE$  is the angle bisector of  $\angle ABC$ .

(7)



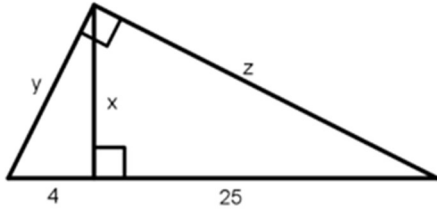
We have  $AB = AC$ ,  $BC = DC$ . Using angle bisector theorem for  $\angle ABC$ , we get that:

$$\frac{AE}{EC} = \frac{AB}{BC} = \frac{AC}{DC} \quad (1) \quad \frac{AE}{EC} = \frac{AF}{FD} \quad (2)(1), (2) \Rightarrow \frac{AF}{FD} = \frac{AC}{DC}$$

From the inverse of the angle bisector theorem we have  $CF$  is the angle bisector of  $\angle ACD$ .

### Exercise Solutions for Pythagorean Theorem:

(1)

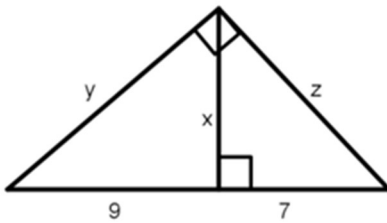


$$x^2 = 4 \times 25 \Rightarrow x = 10$$

$$y^2 = 4 \times 29 \Rightarrow y = 2\sqrt{29}$$

$$z^2 = 29 \times 25 \Rightarrow z = 5\sqrt{29}$$

(2)

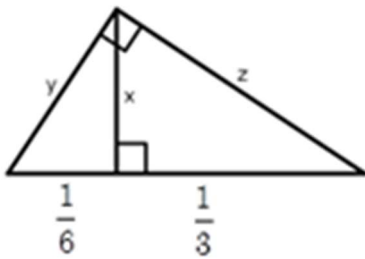


$$x^2 = 7 \times 9 \Rightarrow x = 3\sqrt{7}$$

$$y^2 = 9 \times 16 \Rightarrow y = 12$$

$$z^2 = 7 \times 16 \Rightarrow z = 4\sqrt{7}$$

(3)

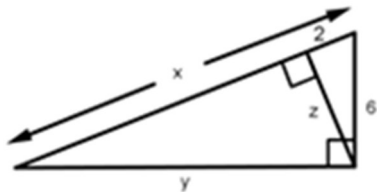


$$x^2 = \frac{1}{3} \times \frac{1}{6} \Rightarrow x = \frac{\sqrt{2}}{6}$$

$$y^2 = \frac{1}{6} \times \frac{1}{2} \Rightarrow y = \frac{\sqrt{3}}{6}$$

$$z^2 = \frac{1}{3} \times \frac{1}{2} \Rightarrow z = \frac{\sqrt{6}}{6}$$

(4)

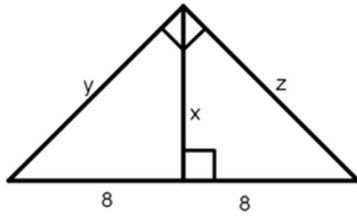


$$6^2 = 2 \times x \Rightarrow x = 18$$

$$y^2 = 18 \times 16 \Rightarrow y = 12\sqrt{2}$$

$$z^2 = 2 \times 16 \Rightarrow z = 4\sqrt{2}$$

(5)

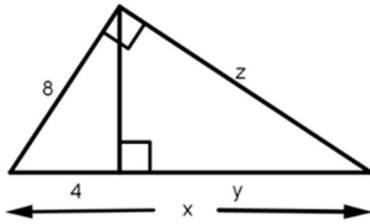


$$x^2 = 8 \times 8 \Rightarrow x = 8$$

$$y^2 = 8 \times 16 \Rightarrow y = 8\sqrt{2}$$

$$z = y = 8\sqrt{2}$$

(6)

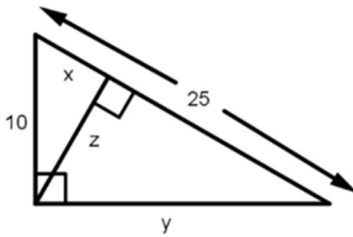


$$8^2 = 4 \times x \Rightarrow x = 16$$

$$y = 16 - 4 \Rightarrow y = 12$$

$$z^2 = 12 \times 16 \Rightarrow z = 8\sqrt{3}$$

(7)



$$10^2 = x \times 25 \Rightarrow x = 4$$

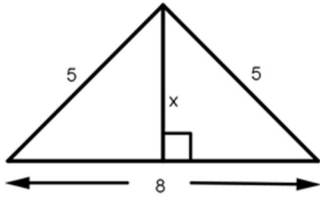
$$y^2 = 21 \times 25 \Rightarrow y = 5\sqrt{21}$$

$$z^2 = 4 \times 21 \Rightarrow z = 2\sqrt{21}$$

(8 – 11)

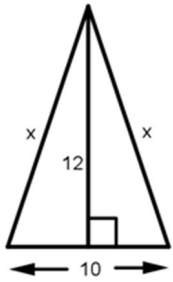
Square diagonal	Square side	Exercise
$2\sqrt{2}$	2	(8)
$10\sqrt{2}$	10	(9)
$20k\sqrt{2}$	$20k$	(10)
$14n$	$7n\sqrt{2}$	(11)

(12)



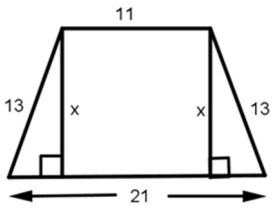
$$x = \sqrt{5^2 - 4^2} = 3$$

(13)



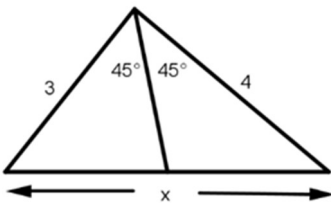
$$x = \sqrt{12^2 + 5^2} = 13$$

(14)



$$x = \sqrt{13^2 - 5^2} = 12$$

(15)



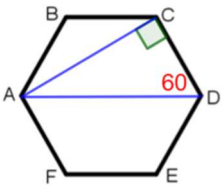
$$x = \sqrt{4^2 + 3^2} = 5$$

## Exercise solutions for Special Triangles:

(1 – 7)

Value of $x$	Exercise Number
$6\sqrt{2}$	1
10	2
$6\sqrt{2}$	3
10	4
6.5	5
$3\sqrt{3}$	6
$10\sqrt{3}$	7

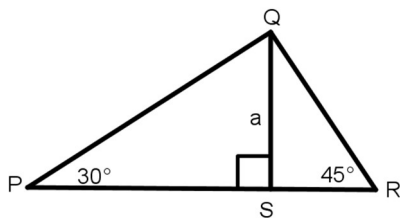
(8)



$$AC = 8\sqrt{3}$$

$$AD = 16$$

(9)



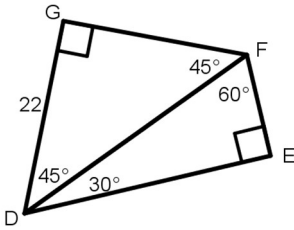
$$QR = a2$$

$$PS = a3$$

$$PQ = 2a$$

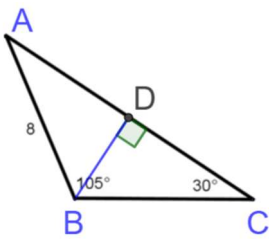
(10)

$$GF = 22DF = 22\sqrt{2}FE = 11\sqrt{2}ED = 11\sqrt{6}$$



(11)

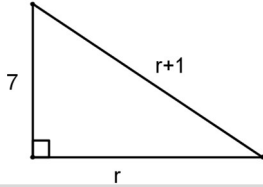
$$\begin{aligned} AB = 8, \quad AD = 4\sqrt{2}, \quad BD = 4\sqrt{2}BC \\ = 8\sqrt{2}, \quad DC = 4\sqrt{6} \Rightarrow \text{Perimeter} \\ = 8 + 12\sqrt{2} + 4\sqrt{6} \end{aligned}$$



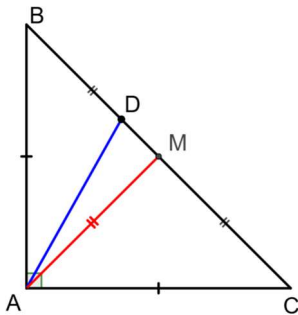
## Challenge Exercises Solutions:

(1)

$$(r + 1)^2 = r^2 + 7^2 \Rightarrow r = 24$$



(2)



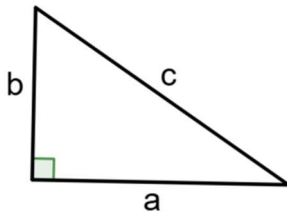
We draw an altitude from  $A$  on the hypotenuse  $BC$ . From the properties of isosceles triangles, the altitude will bisect  $BC$  at  $M$ . Moreover, from the properties of the median in the right angle, we have:

$$\begin{aligned} AM &= CM = BM \\ BD^2 + CD^2 &= (BM - MD)^2 + (CM + MD)^2 \\ &= BM^2 + MD^2 - 2BM \cdot MD + CM^2 + MD^2 \\ &\quad + 2CM \cdot MD = 2(AM^2 + MD^2) = 2AD^2 \end{aligned}$$

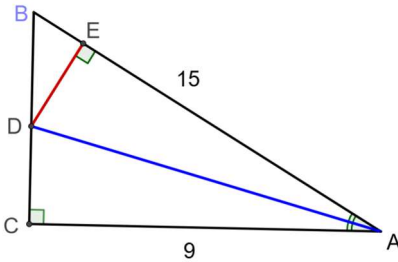
(3)

$$a^2 + b^2 = c^2$$

$$\begin{aligned} \frac{1}{2} a \cdot b = 30 &\Rightarrow a \cdot b = 60 \\ a + b + c = 30 &\Rightarrow a + b = 30 - c \\ \Rightarrow (a + b)^2 &= (30 - c)^2 \Rightarrow a^2 + b^2 + 2a \cdot b \\ &= 900 + c^2 - 60c \Rightarrow c^2 + 2 \cdot 60 = 900 + c^2 - 60c \\ \Rightarrow 120 &= 900 - 60c \Rightarrow c = 13 \\ a \cdot b = 60 &\Rightarrow b = \frac{60}{a} \\ a^2 + b^2 &= c^2 \Rightarrow a^2 + \left(\frac{60}{a}\right)^2 = 13^2 \\ \Rightarrow a^4 - 169a^2 + 3600 &= 0 \\ \Rightarrow (a^2 - 25)(a^2 - 144) &= 0 \Rightarrow a^2 = 25 \\ \Rightarrow a = 5, b = 12 \text{ or } a^2 = 144 &\Rightarrow a = 12, b = 5 \end{aligned}$$



(4)



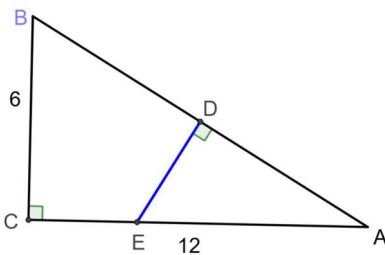
$$BC = \sqrt{225 - 81} = 12$$

$$\frac{AB}{AC} = \frac{BD}{DC} = \frac{3}{5} = \frac{BD}{12 - BD} \Rightarrow BD = 4.5, \quad CD = 7.5$$

$$\triangle ADE \cong \triangle ADC \text{ (ASA)}$$

$$\Rightarrow DE = DC = 4.5$$

(5)



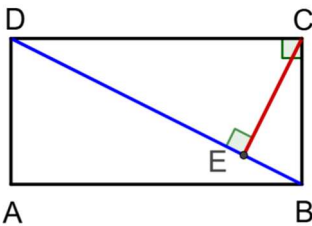
$$AB = \sqrt{144 + 36} = 6\sqrt{5} \Rightarrow AD = DB = 3\sqrt{5}$$

$$\triangle ADE \sim \triangle ACB \text{ (AA)}$$

$$\Rightarrow \frac{AD}{AC} = \frac{AE}{AB} \Rightarrow \frac{3\sqrt{5}}{12} = \frac{AE}{6\sqrt{5}} \Rightarrow AE = 7.5$$

$$\Rightarrow CE = 12 - 7.5 \Rightarrow CE = 4.5$$

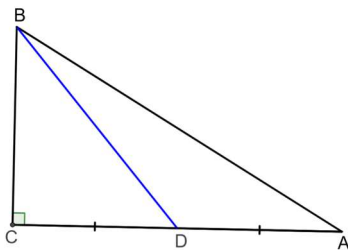
(6)



$$CE^2 = BE \cdot DE \Rightarrow 25 = \frac{1}{4}BD \cdot \frac{3}{4}BD$$

$$\Rightarrow BD = \frac{20\sqrt{3}}{3}$$

(7)

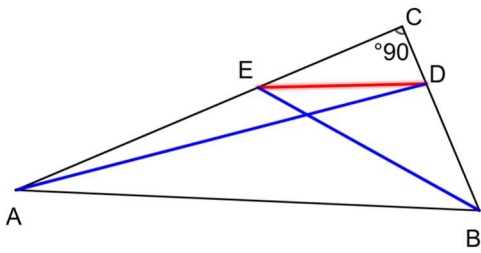


$$AB = 2CD$$

$$AB^2 + 3BC^2 = BC^2 + (2CD)^2 + 3BC^2$$

$$= 4(BC^2 + CD^2) = 4BD^2$$

(8)



$$(1) \{BC^2 + CE^2 = BD^2 \quad AC^2 + CD^2 = AD^2$$

$$(2) \{CD^2 + CE^2 = ED^2 \quad AC^2 + BC^2 = AB^2$$

By comparing the sum of (1) and the sum (2) We find that:

$$AD^2 + BE^2 = AB^2 + DE^2$$

## Number theory Solutions

### Solutions for Revision Exercises:

(1)

- 73 is a prime number.
- 91 is a composite number since  $91 = 7 \times 13$ .
- 101 is a prime number.
- 143 is a composite number since  $143 = 11 \times 13$ .
- 199 is a prime number.

(2)

From the law of the number of divisors we get:

- a)  $(1 + 1)(1 + 1) = 4$
- b)  $(2 + 1)(1 + 1) = 6$
- c)  $(2 + 1)(2 + 1) = 9$
- d)  $(m + 1)(n + 1) = mn + m + n + 1$

(3)

Notice that in any three consecutive integers. We have one even and one that is divisible by 3 (they could be the same number). Thus, the multiplication must be divided by 6.

(4)

**(a)** We factorize

$$30 = 2 \times 3 \times 5.$$

We need to prove that the product is divisible by **2, 3, and 5** simultaneously.

- **Divisibility by 5:** In any sequence of five consecutive natural numbers, exactly one number must be divisible by 5.
- **Divisibility by 3:** In any sequence of three consecutive natural numbers, exactly one number must be divisible by 3. Since we have five numbers, this condition is satisfied.
- **Divisibility by 2:** In any sequence of two consecutive natural numbers, exactly one number must be divisible by 2. Since we have five numbers, this condition is also satisfied.

Since the product is divisible by **2, 3, and 5**, it is divisible by

$$2 \times 3 \times 5 = 30.$$

**(b)** Notice that it is enough to show that the multiplication is divisible by 120 (since 30 divides 120). The factorization of 120 is:

$$120 = 2^3 \times 3 \times 5$$

Thus, we need to show that the number is divisible by 3,5,8. Similar to the previous question, notice that any five consecutive integers must have a multiple of 3 and a multiple of 5. Therefore, we only need to show that the multiplication is also a multiple of 8. Notice that in any 4 consecutive numbers, we have two even integers. Moreover, one of them must be a multiple of 4. Thus the multiplication must be divisible by 120.

(5)

By factorizing 660, we get:

$$660 = 4 \times 3 \times 5 \times 11.$$

Thus, we need to make sure that  $n!$  contains the primes 2,3,5,11. This means that  $n \geq 11$ . By substituting  $n = 11$  it is clear that 660 divides 11!

(6)

The number of consecutive zeros at the end of any number depends on how many times does 10 divide the number. However, notice that  $10 = 2 \times 5$ . Thus, we need to look for the number of 5's and 2's in  $10!$ . We can clearly see that 5 only appears twice at 5,10, while 2 appears more than twice. This means there is exactly two 10's that divide  $10!$ . This gives an answer of 2.

(7)

Not possible. Notice that  $24!$  ends with exactly four 0's while  $25!$  ends with exactly six 0's. And any number  $n > 25$  will have that  $n!$  has at least six 0's. On the other hand, any number  $m < 24$  will have that  $m!$  has at most four 0's.

(8)

By factorizing the left hand side we get that:

$$(x - y)(x + y) = 33 = 3 \times 11$$

Since the divisors of 33 are either  $3 \times 11$  or  $1 \times 33$ . Then, we have two cases:

Case 1:  $x - y = 3, x + y = 11$ . This gives:

$$2x = 14 \rightarrow x = 7$$

$$y = 7 - 3 \rightarrow y = 4$$

$$(x, y) = (7, 4).$$

Case 2:  $x - y = 1, x + y = 33$ . This gives:

$$2x = 34 \rightarrow x = 17$$

$$y = 33 - 17 \rightarrow y = 16$$

$$(x, y) = (17, 16).$$

## Solutions for $gcd$ Exercises:

(1)

By factorizing both 8 and 9, we get that:

$$8 = 2^3, 9 = 3^2$$

And since they do not contain any common factors, we get that  $gcd(8,9) = 1$ .

(2)

By factorizing both 54 and 96, we get that:

$$54 = 2 \times 3^3, 96 = 2^5 \times 3$$

We take the common factors with the smaller power, we get that  $gcd(54,96) = 2^1 \times 3^1$ .

(3)

By factorizing both 35 and 91, we get that:

$$35 = 5 \times 7, 91 = 7 \times 13$$

We take the common factors with the smaller power, we get that  $gcd(35,91) = 7^1$ .

(4)

By factorizing both 54 and 6, we get that:

$$54 = 2 \times 3^3, 6 = 2 \times 3$$

We take the common factors with the smaller power, we get that  $gcd(54,6) = 2^1 \times 3^1$ .

(5)

By factorizing both 256 and 199, we get that:

$$256 = 2^8, 199 = 199$$

And since they do not contain any common factors, we get that  $gcd(256,199) = 1$ .

(6)

By factoring the number 120. We get that:

$$120 = 2^3 \times 3 \times 5$$

We will check which of the following choices and check which ones satisfy that

$\gcd(n, 120) = 24$ :

- (a)  $\gcd(120, 2 \times 3^3) = 6$
- (b)  $\gcd(120, 2^2 \times 3^3) = 12$
- (c)  $\gcd(120, 2^3 \times 3^2 \times 11) = 24$
- (d)  $\gcd(120, 2^4 \times 3^3 \times 5) = 120$

Thus, the correct answer is (C).

## Solutions for *lcm* Exercises:

(1)

By factorizing both 8 and 9, we get that:

$$8 = 2^3, 9 = 3^2$$

By considering all the factors with the higher power, we get that  $lcm(8,9) = 2^3 \times 3^2$ .

(2)

By factorizing both 54 and 96, we get that:

$$54 = 2 \times 3^3, 96 = 2^5 \times 3$$

By considering all the factors with the higher power, we get that  $lcm(54,96) = 2^5 \times 3^3$ .

(3)

By factorizing both 35 and 91, we get that:

$$35 = 5 \times 7, 91 = 7 \times 13$$

By considering all the factors with the higher power, we get that  $lcm(35,91) = 5 \times 7 \times 13$ .

(4)

By factorizing both 54 and 6, we get that:

$$54 = 2 \times 3^3, 6 = 2 \times 3$$

By considering all the factors with the higher power, we get that  $lcm(6,54) = 2 \times 3^3$ .

(5)

By factorizing both 256 and 199, we get that:

$$256 = 2^8, 199 = 199$$

By considering all the factors with the higher power, we get that  $lcm(256,199) = 2^8 \times 199$ .

(6)

By factoring the number 120. We get that:

$$120 = 2^3 \times 3 \times 5$$

We will check which of the following choices and check which ones satisfy that

$$lcm(n, 20) = 180:$$

- (a)  $lcm(20, 2 \times 3^3) = 2^2 \times 3^3 \times 5 = 540$
- (b)  $lcm(20, 2^2 \times 3^2) = 2^2 \times 3^2 \times 5 = 180$
- (c)  $lcm(20, 2^3 \times 3^2) = 2^3 \times 3^2 \times 5 = 360$
- (d)  $lcm(20, 2^4 \times 3^3 \times 5) = 2^4 \times 3^3 \times 5 = 2160$

Thus, the correct answer is (b).

## Solutions for Perfect Squares Exercises:

(1)

From the important rule, we get that:

$$n \times 18 = 6 \times 36 \rightarrow n = 12$$

(2)

By factorizing the numbers we get that:

- $196 = 2^2 \times 7^2$
- $192 = 2^6 \times 3$
- $169 = 13^2$
- $240 = 2^4 \times 3 \times 5$

Thus, the perfect squares are 169,196.

(3)

Notice that the sum of the digits of the number is:

$$0 \times 100 + 1 \times 100 + 2 \times 100 = 300.$$

From the divisibility rule by 3 and 9. We see that the number is divisible by 3 but not 9. Thus, the power of 3 that divides it is 1 which is odd. Therefore, the number cannot be a perfect square.

(4)

From the rule of the number of divisors. We have that the number of divisors of  $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_n^{\alpha_n}$  is:

$$(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_n + 1)$$

If the number of divisors is odd. This means that all of the numbers in the brackets are odd.

(Since if one of them is even we have  $even \times odd = even$ ). This means that:

$$(\alpha_1 + 1) = odd, (\alpha_2 + 1) = odd, \dots, (\alpha_n + 1) = odd$$

And this means that all of  $\alpha_k$  have to be even since  $\alpha_k = odd - 1 = even$ . Thus, the number is a perfect square.

(5)

Let us write the prime factorizations of both  $a, b$ . We have that:

$$a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_n^{\alpha_n}, b = p_1^{\beta_1} \cdot p_2^{\beta_2} \dots p_n^{\beta_n}$$

Where  $\alpha_k, \beta_k \geq 0$ . Then, the rule of the greatest common divisor gives us:

$$\gcd(a, b) = p_1^{\min(\alpha_1, \beta_1)} \cdot p_2^{\min(\alpha_2, \beta_2)} \dots p_n^{\min(\alpha_n, \beta_n)}$$

on the other hand. The rule of the least common multiple gives us:

$$\text{lcm}(a, b) = p_1^{\max(\alpha_1, \beta_1)} \cdot p_2^{\max(\alpha_2, \beta_2)} \dots p_n^{\max(\alpha_n, \beta_n)}$$

However, it is not hard to see that:

$$(\alpha_k, \beta_k) + (\alpha_k, \beta_k) = \alpha_k + \beta_k$$

(Since one of them is max and the other is the min)

Thus, we have:

$$\begin{aligned} d \times m &= p_1^{\min(\alpha_1, \beta_1)} \cdot p_2^{(\alpha_2, \beta_2)} \dots p_n^{(\alpha_n, \beta_n)} \times p_1^{\max(\alpha_1, \beta_1)} \cdot p_2^{(\alpha_2, \beta_2)} \dots p_n^{(\alpha_n, \beta_n)} \\ &= p_1^{\min(\alpha_1, \beta_1) + \max(\alpha_1, \beta_1)} \cdot p_2^{(\alpha_2, \beta_2) + \max(\alpha_2, \beta_2)} \dots p_n^{(\alpha_n, \beta_n) + \max(\alpha_n, \beta_n)} \\ &= p_1^{\alpha_1 + \beta_1} \cdot p_2^{\alpha_2 + \beta_2} \dots p_n^{\alpha_n + \beta_n} \end{aligned}$$

On the other hand, we have:

$$a \times b = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_n^{\alpha_n} \times p_1^{\beta_1} \cdot p_2^{\beta_2} \dots p_n^{\beta_n} = p_1^{\alpha_1 + \beta_1} \cdot p_2^{\alpha_2 + \beta_2} \dots p_n^{\alpha_n + \beta_n}$$

Therefore, we have  $md = ab$  which is our desired result.

## Solutions (Combinatorics)

### Combinations

(1)

$${}^5C_2 = 10$$

(3)

The case where the number of  $E$ 's is greater than the number of  $F$ 's means  $E > 5$ .

So, the number of  $E$ 's can be 6, 7, 8, 9, or 10.

The total number of ways is:

$${}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \\ 210 + 120 + 45 + 10 + 1 = 386 \text{ words.}$$

(5)

$$a) {}^8C_3 \cdot {}^5C_3 = 560$$

$$b) \frac{{}^8C_4 \cdot {}^4C_4}{2} = 35$$

$$c) \frac{{}^8C_2 \cdot {}^6C_2 \cdot {}^4C_2 \cdot {}^2C_2}{4!} = 105$$

$$d) \frac{{}^8C_4 \cdot {}^4C_2 \cdot {}^2C_2}{2!} = 210$$

$$e) {}^8C_4 \cdot {}^4C_3 \cdot {}^1C_1 = 280$$

(2)

The number of letters other than  $E$  is seven, which can be arranged in  $7!$  ways.

There are  $8$  gaps between them where the three  $E$ 's can be placed, with no two  $E$ 's adjacent. We choose  $3$  out of  $8$  positions (order doesn't matter).

$$7! \cdot {}^8C_3 = 282240$$

(4)

Between the four letters:  ${}^4C_2 = 6$

Between the six odd numbers:  ${}^6C_2 = 15$

Between the three even numbers:  ${}^3C_2 = 3$

Total =  $6 + 15 + 3 = 24$  chords.

(6)

The total number of people including Osama is 11.

Osama sits in a fixed position, with one friend on his right and another on his left. The number of pairs of friends who can sit next to him = number of ways to choose 2 out of 10 friends:

$${}^{10}C_2 = 45 \text{ different pairs.}$$

## Exercises on Counting Principles

(1)

We choose one person for the first room, then two people for the second room, and finally the remaining students for the last room.

(Note that the last combination can be ignored, since the remaining four students will necessarily go to the last room.)

$$\binom{7}{1} \times \binom{6}{2} \times \binom{4}{4} = 105$$

(3)

We consider the possible cases for the number of B's and add them together:

$$\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \binom{8}{3}$$

(2)

All 10-digit numbers:  $9 \times 10^9$

Those with all digits different:  $9 \times 9 \times 8 \times 7 \times \dots \times 1$

Therefore, the required number is:  $9 \times 10^9 - 9 \times 9!$

(4)

The total number is:  $9 \times 10^6$

Those that do not contain the digit "1" =  $8 \times 9^6$

This is not equal to half, therefore the answer is: No.

**(5)**

We look at the first 9 digits from the left.

If their sum is even (call it AAA), then the

units digit must be even and has 5

choices. If their sum is odd (call it BBB),

then the units digit must be odd and has

5 choices.

So the total is:  $5A + 5B = 5(A + B)$

And  $A + B$  is the number of 9-digit

numbers, which equals  $9 \times 10^8$

Final answer:  $5 \times 9 \times 10^8$

**(7)**

We treat A and B as one block together

with the remaining four books, so there

are  $5!$  arrangements. Inside the block,

there are two possible orders (AB or BA):

$$5! \times 2$$

**(6)**

We choose a row and a column for each

rook. No row or column is repeated, since

two rooks cannot be in the same row or

column.

Then we divide by  $4!$  because the rooks are

identical: 
$$\frac{(8 \times 8) \times (7 \times 7) \times (6 \times 6) \times (5 \times 5)}{4!}$$

**(8)**

Since Jawad is fixed in the committee, we

chose two people from the remaining nine:

$$\binom{9}{2}$$

## Ascending and Descending Order

(1)

Note that choosing 3 digits is enough to determine whether the order is ascending or descending. In the ascending case, zero cannot be chosen because it would occupy the hundreds place.

Ascending: choose 3 different digits from

$$1, 2, \dots, 9: \binom{9}{3}$$

Descending: choose 3 different digits

$$\text{from } 0, 1, \dots, 9: \binom{10}{3}$$

$$\text{So the total number is: } \binom{9}{3} + \binom{10}{3} = 204$$

(3)

*In the ascending case, the solution is different because zero cannot be chosen:*

$$\binom{8}{4} \quad (a)$$

$$\binom{9}{4} \quad (b)$$

(2)

Same idea as the previous question, so the answers for the two parts are equal:

Which is the same as:

$$\binom{8}{5} = \binom{8}{3} = 56$$

(4)

*The number of ways to draw 3 cards with order is:  $10 \times 9 \times 8$*

$$\text{Salem wins (ascending order): } \binom{10}{3} = 120$$

*Since each set of 3 cards has only one ascending arrangement, Mohammed has a higher chance of winning.*

*Result: Mohammed wins in about 16.7% of the cases.*

## Counting Paths on a Grid

(1)

We need 6 forward jumps and 7 backward jumps to reach  $-1$ . The number of ways is the number of ways to choose 6 forward jumps out of 13 jumps:

$$\binom{13}{6} \text{ Or } \binom{13}{7} \text{ and they are equal.}$$

(3)

Each rectangle is determined by choosing two vertical lines and two horizontal

$$\text{lines: } \binom{10}{2} \times \binom{8}{2} = 1260$$

(5)

There are 3 vowels (A, O, U). We choose 2

$$\text{of them: } \binom{3}{2} = 3$$

There are 6 consonants: M, T, T, H, C, N.

- 3 consonants with no T:  $\binom{4}{3} = 4$
- 3 consonants with one T:  $\binom{4}{2} = 6$

(2)

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(4)

We have 25 letters. We choose two of them using combinations (and they have only one order since they are alphabetically ordered).

We also choose two digits from 9 digits:

$$\binom{25}{2} \times \binom{9}{2} = 10800$$

- 3 consonants with two T's:  $\binom{4}{1} = 4$

Since in all cases there are 4 non-T letters,

$$\text{the total is: } 3(4 + 6 + 4) = 42$$



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