وزارة التعليم
Ministry of Education

موهبة
Mawhiba

نسمو
NSMO

National Science and Mathematics Olympiad

Learning Materials for the Fourth Stage Finals of "NSMO"2026

Mathematics - Senior

بسم الله الرحمن الرحيم

Table of Contents

Introduction

Our outstanding sons and daughters,

We congratulate you on reaching the **pre-final stage** of the National Olympiad for Science and Mathematics, a stage that represents the culmination of your continuous efforts in understanding, training, and creativity.

In this special module, we will continue to deepen our exploration of the four branches: **Combinatorics, Geometry, Algebra, and Number Theory**, with topics such as combinations, the Pythagorean theorem, quadratic equations, and common factors and multiples.

This stage aims to refine your **higher-level thinking skills** and **logical analysis**, and to train you to deal with complex problems that require precision and advanced reasoning.

It also serves as a direct preparation for the **final stage** of the competition, where true excellence is demonstrated through the ability to connect mathematical concepts and apply them in new situations.
We are confident that you are worthy of this stage, and we ask God to grant you success and guidance on your

The Scientific Team for the National Science and Mathematics Olympiad (NSMO) – Mathematics Track

First Unit: ALGEBRA

Quadratic Equations

The Expression: $ax^2 + bx + c = 0$

(Where a, b, c are constants and $a \neq 0$) is called a **quadratic equation**. Its solutions are called **roots** or **zeros** of the polynomial.

There are several methods to find the roots of the equation $ax^2 + bx + c = 0$, including:

1. Factoring.
2. Completing the square.
3. The quadratic formula.

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

where the discriminant Δ is given by: $\Delta = b^2 - 4ac$

- If $\Delta > 0$: the equation has **two distinct real roots**.
- If $\Delta < 0$: the equation has **no real roots**.
- If $\Delta = 0$: the equation has **two equal real roots**.

Problem 1:

Solve the equation: $x^2 + 6x + 5 = 0$

in three different ways.

Solution:

1- Factoring.

$$\boldsymbol{x^2 + 6x + 5 = 0}$$

We look for two numbers whose product is 5 and whose sum is 6.

$\Longrightarrow \boldsymbol{x^2 + 6x + 5 = (x + 1)(x + 5) = 0}$
$\Longrightarrow \boldsymbol{(x + 1) = 0} \; or \; \boldsymbol{(x + 5) = 0}$
$\Longrightarrow \boldsymbol{x = -1} \; or \; \boldsymbol{x = -5}$

2- Completing the square.

$$\boldsymbol{x^2 + 6x + 5 = 0}$$

Move the constant term to the right side and add the square of half the coefficient of $\boldsymbol{x}$ to both sides.

$\Longrightarrow \boldsymbol{x^2 + 6x + 3^2 = -5 + 3^2}$
$\Longrightarrow \boldsymbol{(x + 3)^2 = 4}$

Taking the square root of both sides:

$\Longrightarrow \boldsymbol{x + 3} = \pm 2$
$\Longrightarrow \boldsymbol{x = -1} \; or \; \boldsymbol{x = -5}$

3- The quadratic formula.

$$\boldsymbol{x^2 + 6x + 5 = 0}$$

$a = 1, b = 6, c = 5 \;\; \Longrightarrow \Delta = b^2 - 4ac = 6^2 - 4(1)(5) = 36 - 20 = 16$

$x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$\Longrightarrow x = \dfrac{-6 \pm \sqrt{16}}{2(1)}$

$\Longrightarrow x = \dfrac{-6 + 4}{2} \;\; or \;\; x = \dfrac{-6 - 4}{2}$

$\Longrightarrow \boldsymbol{x = -1} \; or \; \boldsymbol{x = -5}$

Relationship Between Roots and Coefficients of a Quadratic Equation:

This is known as "**Vieta's Formulas**". If r, s are the roots of the quadratic equation:

$$ax^2 + bx + c = 0$$

Then the **sum of the roots** is given by the relation: $r + s = \frac{-b}{a}$

And the **product of the roots** is given by the relation: $rs = \frac{c}{a}$.

Exercises:

(1) Find all solutions to each of the following equations:

$(a)\ x^2 - 12x - 540 = 0$
$(b)\ 3x^2 = 10x + 24$
$(c)\ (x^4 - 11x^3 + 24x^2) - (4x^2 - 44x + 96) = 0$

(2) Find the value of a such that the equation:

$$ax^2 - 5x + 9 = 0$$

has exactly one real root.

(3) How many integers x satisfy the equation:

$$(x^2 - x - 1)^{x+2} = 1$$

(4) If one of the roots of the equation:

$$a(b-c)x^2 + b(c-a)x + c(a-b) = 0 \text{ is } x = 1.$$

find the other root in terms of a, b and c.

(5) if a and b are the roots of the quadratic equation $x^2 - mx + 2 = 0$.

and given that $a + \frac{1}{b}, b + \frac{1}{a}$ are the roots of the quadratic equation $x^2 - px + q = 0$ find the value of q.

(6) Find all solutions for the following system of equations:

$$\begin{cases} x^2 + xy = 39 \\ x - y = -33 \\ y + z = 13 \end{cases}$$

(7) if $p(x)$ is a second-degree polynomial such that

$$p(0) = -1, p(1) = 9, p(2) = 25$$

Find $p(-1)$.

(8) Find all values of k such that the equation:

$$x^2 + kx + 27 = 0$$

has two distinct real roots.

(9) Prove that if:

$$\frac{a+b}{a}=\frac{b}{a+b}$$

then it is impossible for both $a\ and\ b$ to be real numbers.

(10) Find the real solutions for the equation:

$$(2+(2+(2+(2+x)^2)^2)^2)^2=15129$$

Given that:

$(15129=123^2)$

(11) Find all real solutions (x, y) that satisfy the system:

$$x^2+y=12=y^2+x$$

(12) Solve the following system of equations in the set of real numbers:

$$\begin{cases} 2x_1=x_5^2-23 \\ 4x_2=x_1^2+7 \\ 6x_3=x_2^2+14 \\ 8x_4=x_3^2+23 \\ 10x_5=x_4^2+34 \end{cases}$$

Other Forms of Identities

Definition:

An identity is an equality between two equivalent expressions, where algebraic operations can be applied to one side to derive the other. This contrasts with an equation, which is an equality between two expressions where the goal is to find the specific value of the variable that makes both sides equal.

The significance of identities, particularly special product identities, lies in their utility for simplifying various algebraic expressions and solving numerous equations. In this section, we will use the identities previously studied to prove and establish other forms of algebraic identities.

Problem 1:

Using the identities for the square of a binomial sum and the square of a binomial difference, deduce the other side of the following identity:

$(a+b)^2+(a-b)^2=$ ....

Solution:

Using the identity for the square of a binomial sum, the expression $(\boldsymbol{a}+\boldsymbol{b})^2$ can be expanded as follows:

$$(\boldsymbol{a}+\boldsymbol{b})^2=\boldsymbol{a}^2+\boldsymbol{b}^2+2\boldsymbol{a}\boldsymbol{b}$$

Using the identity for the square of a binomial difference, the expression $(\boldsymbol{a}-\boldsymbol{b})^2$ can be expanded as follows:

$$(\boldsymbol{a}-\boldsymbol{b})^2=\boldsymbol{a}^2+\boldsymbol{b}^2-2\boldsymbol{a}\boldsymbol{b}$$

By adding the two results, we obtain the required expression:

$$(\boldsymbol{a}+\boldsymbol{b})^2+(\boldsymbol{a}-\boldsymbol{b})^2=2\boldsymbol{a}^2+2\boldsymbol{b}^2$$

Exercises:

(1) Using the identities you have studied previously, deduce the missing side of the identity in each of the following:

(a) $(a+b)^2-(a-b)^2=\cdots$

(b) $\ldots=(a-b)^3+3ab(a-b)$

(c) $\ldots=(a+b)^3-3ab(a+b)$

(d) $\ldots=\sqrt{(a+b)^2-4ab}$

(2) Challenge: Using a similar approach, deduce the missing side of the following:

$a)$ $a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]$

$b)$ $(a+b+c)^3=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(3) If $\boldsymbol{x}$ is a real number such that $\sqrt[3]{\boldsymbol{x}}+\frac{\mathbf{1}}{\sqrt[3]{x}}=\mathbf{3}$, find the value of $\boldsymbol{x}^{\mathbf{3}}+\frac{\mathbf{1}}{\boldsymbol{x}^{\mathbf{3}}}$

(4) Solve the equation: $\sqrt[3]{1+\sqrt{x}}+\sqrt[3]{1-\sqrt{x}}=2$

(5) If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ are real numbers satisfying:

$(a-b)(a+b-c)=3\,,(b-c)(b+c-a)=5$

Find the value of: $(c-a)(a+c-b)$.

(6) Solve the following equation:

$$\sqrt{x-2}+\sqrt{4-x}=\sqrt{6-x}$$

(7) Solve the following equation:

$$ab(x^2+1)=(a^2+b^2)x$$

given that $\boldsymbol{ab}\neq\mathbf{0}$ and $\boldsymbol{a}>\boldsymbol{b}$.

(8) Solve the equation:

$$\sqrt{x-10}-\frac{6}{\sqrt{x-10}}=5$$

(9) Solve the equation:

$$\frac{(2x-1)^2}{2}+\frac{(3x-1)^2}{3}+\frac{(6x-1)^2}{6}=1$$

Factoring

3-1 Factoring Using Identities - Adding and Subtracting Terms:

Factoring is an effective method for rewriting certain algebraic expressions—whether sums or differences—into products, which makes them easier to work with.

Some expressions can be factored by using well-known identities, such as the difference of two squares identity:

$$a^2 - b^2 = (a-b)(a+b)$$

This identity can be proved in either of the following two ways:

First method: Expand the right-hand side by multiplying the binomials, then simplify to obtain the left-hand side directly.

Second method: Start from the left-hand side, add and subtract the term $\boldsymbol{ab}$, then factor by grouping as follows:

$$\begin{aligned} a^2 - b^2 &= a^2 - b^2 + ab - ab \\ &= a^2 + ab - b^2 - ab \\ &= a(a+b) - b(a+b) \\ &= (a+b)(a-b) \end{aligned}$$

Problem 1:

Factor:

$$a^3 - b^3$$

Solution:

$$\begin{aligned} a^3 - b^3 &= (a-b)^3 + 3ab(a-b) \\ &= (a-b)[(a-b)^2 + 3ab] \\ &= (a-b)(a^2 - 2ab + b^2 + 3ab) \\ &= (a-b)(a^2 + ab + b^2) \end{aligned}$$

General Formulas:

1) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})$

2) $x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + y^{n-1})\ \ for\ odd\ n \in \mathbb{N}$

Exercises:

Factor each of the following expressions:

$(1)\ a^3 + b^3$
$(2)\ x^2 - (a+b)x + ab$
$(2)\ x^2 + (a+b)x + ab$
$(4)\ x^4 + x^2 + 1$
$(5)\ x^4 + 2x^3 + 2x^2 + 2x + 1$
$(6)\ x^5 + x + 1$
$(7)\ (x+y)(x-y) + 4(y-1)$
$(8)\ x^3(x-2y) + y^3(2x-y)$
$(9)\ x^2y - y^2z + z^2x - x^2z + y^2x + z^2y - 2xyz$
$(10)\ 1 + a + b + c + ab + bc + ca + abc$
$(11)\ (ax+by)^2 + (ay-bx)^2 + c^2x^2 + c^2y^2$

(12) If

$$(c + \frac{1}{c} + 1)(c + \frac{1}{c}) = 1$$

find

$$(3c^{100} + \frac{2}{c^{100}} + 1)(c^{100} + \frac{2}{c^{100}} + 3)$$

2-3 Factoring the Expression $(a^3 + b^3 + c^3 - 3abc)$:

The expression $\boldsymbol{a^3 + b^3 + c^3 - 3abc}$ is one of the well-known expressions in Olympiad competitions, due to the large number of problems in which it appears.

This expression can be factored using the method of **adding and subtracting terms**, followed by **factoring by grouping**, as follows:

$$a^3 + b^3 + c^3 - 3abc = a^3 + b^3 + c^3 - 3abc +$$
$$ab^2 - ab^2 + ac^2 - ac^2 +$$
$$a^2b - a^2b + a^2c - a^2c +$$
$$bc^2 - bc^2 + b^2c - b^2c =$$
$$a^3 + ab^2 + ac^2 - a^2b - abc - a^2c +$$
$$b^3 + a^2b + bc^2 - ab^2 - bc^2 - abc +$$
$$c^3 + a^2c + b^2c - abc - bc^2 - ac^2 =$$
$$a(a^2 + b^2 + c^2 - ab - bc - ca) +$$
$$b(a^2 + b^2 + c^2 - ab - bc - ca) +$$
$$c(a^2 + b^2 + c^2 - ab - bc - ca) = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

Hence, the final factorization is:

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

It can also be written in the equivalent form:

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)[a^2 + b^2 + c^2 - ab - bc - ca]$$
$$= \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

Important Note:

In the special case where $\boldsymbol{a + b + c = 0}$, we have $\boldsymbol{a^3 + b^3 + c^3 = 3abc}$

This formula is used to solve many problems.

Exercises:

(1) If:

$$a + b = 5\ , \qquad b + c = 10\ , \qquad c + a = 15$$

find

$$(a + b + c)^3 - a^3 - b^3 - c^3$$

(2) Solve the equation:

$$(x - 2009)^3 + (x - 2010)^3 + (x - 2011)^3 = 3(x - 2009)(x - 2010)(x - 2011)$$

(3) If:

$$a = 2009\ , b = 2010\ , c = \frac{1}{2010}$$

find

$$(a + b + c)^3 - (a + b - c)^3 - (c + b - a)^3 - (a - b + c)^3 - 23abc$$

(4) Solve the equation:

$$\sqrt[3]{x - 3} + \sqrt[3]{x + 3} - \sqrt[3]{x} = 0$$

Vieta's Formulas

They are relations that connect the roots of a polynomial to its coefficients, where each coefficient corresponds to a symmetric sum of the polynomial's roots.

At the beginning, we will discuss special cases of these relations, and then we will deduce the general form from them.

Definition:

We say that a is a root (zero) of the polynomial $f(x)$ if $f(a) = 0$.

Example:

The polynomial $f(x) = x^2 + 5x + 6$ has the roots -3 and -2, since

$$f(-2) = f(-3) = 0$$

4-1 Vieta's Relation for a Quadratic Polynomial:

If r_1 and r_2 are the two roots of the quadratic polynomial $f(x) = a_2x^2 + a_1x + a_0$, $(a_2 \neq 0)$ then $f(x)$ can be written in factored form as:

$$\begin{aligned} f(x) &= a_2(x - r_1)(x - r_2) \\ &= a_2(x^2 - xr_1 - xr_2 + r_1r_2) \\ &= a_2(x^2 - (r_1 + r_2)x + r_1r_2) \\ &= a_2x^2 - a_2(r_1 + r_2)x + a_2r_1r_2 \end{aligned}$$

By comparing coefficients in the two expressions, we find that the sum of the roots is given by:

$$-a_2(r_1 + r_2) = a_1 \quad \Longrightarrow \quad r_1 + r_2 = \frac{-a_1}{a_2}$$

And the product of the roots is given by:

$$a_2r_1r_2 = a_0 \quad \Longrightarrow \quad r_1r_2 = \frac{a_0}{a_2}$$

4-2 Vieta's Relations for a Cubic Polynomial:

If r_1, r_2, r_3 are the roots of the cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, $(a_3 \neq 0)$ then $f(x)$ can be written in factored form as:

$$f(x) = a_3(x - r_1)(x - r_2)(x - r_3) =$$
$$a_3(x^3 - x^2r_1 - x^2r_2 - x^2r_3 + r_1r_2x + r_2r_3x + r_3r_1x + r_1r_2r_3) =$$
$$a_3(x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_2r_3 + r_3r_1)x + r_1r_2r_3$$
$$= a_3x^3 - a_3(r_1 + r_2 + r_3)x^2 + a_3(r_1r_2 + r_2r_3 + r_3r_1)x - a_3r_1r_2r_3$$

By comparing coefficients in the two expressions, we obtain:

Sum of the roots:

$$-a_3(r_1 + r_2 + r_3) = a_2 \quad \Longrightarrow \quad r_1 + r_2 + r_3 = \frac{-a_2}{a_3}$$

Sum of the pairwise products of the roots:

$$a_3(r_1r_2 + r_2r_3 + r_3r_1) = a_1 \quad \Longrightarrow \quad r_1r_2 + r_{2r_3} + r_3r_1 = \frac{a_1}{a_3}$$

Product of the roots:

$$-a_3r_1r_2r_3 = a_0 \quad \Longrightarrow \quad r_1r_2r_3 = \frac{-a_0}{a_3}$$

Problem 1:

If r_1, r_2, r_3 are the roots of the equation $5x^3 - 2x + 3 = 0$ write Vieta's relations for this equation.

Solution:

Sum of the roots: $r_1 + r_2 + r_3 = 0$

Sum of the pairwise products of the roots: $r_1r_2 + r_2r_3 + r_3r_1 = \frac{-2}{5}$

Product of the roots: $r_1r_2r_3 = \frac{-3}{5}$

Exercises:

(1) If a and b are the roots of the polynomial $2x^2 - 3x + m = 0$ and $a = 2b$, find the value of m.

(2) Let $f(x) = x^3 - 5x^2 + 12x - 19$

If a, b, c are the roots of $f(x)$, find:

$i)\ a^2 + b^2 + c^2$

$ii)\ \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

(3) Find the cubic polynomial whose roots are the numbers $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, and whose coefficient of x^3 is 1, given that:

$$abc = -64$$

$$a^2 + b^2 + c^2 = 84$$

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{-3}{32}$$

(4) Let α, β, and δ be the real roots of the cubic equation $x^3 + 3x^2 - 24x + 1 = 0$

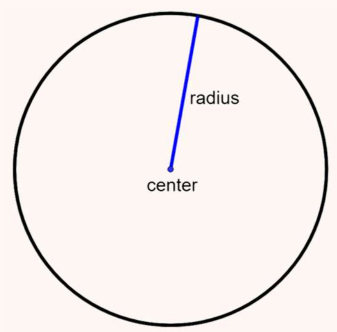
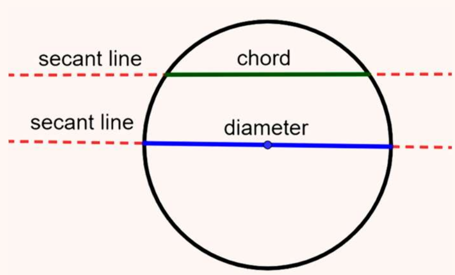
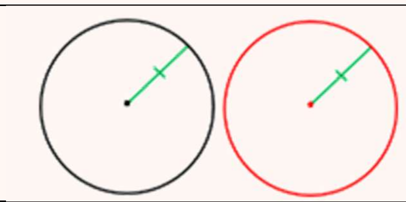
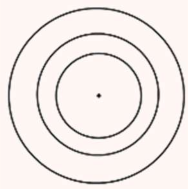
Show that:

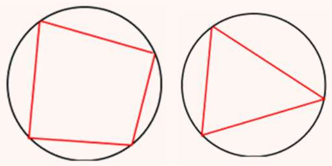
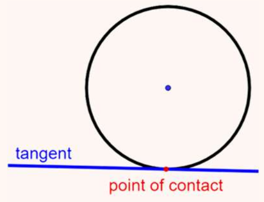
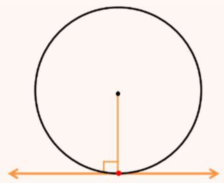
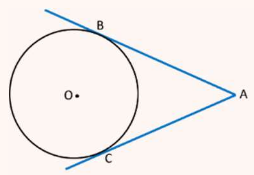
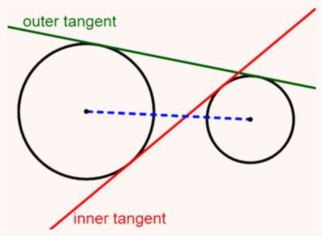
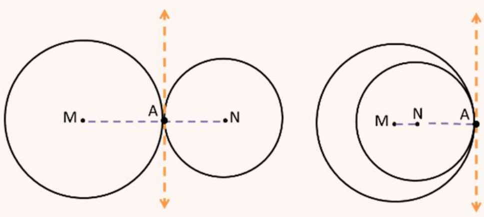
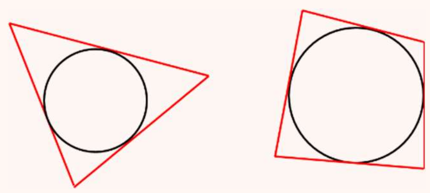
$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\delta} = 0$$

Second Unit: Geometry

Circles

In this part of geometry, we will study the topic of the circles, which is of great importance in completing our knowledge of the topics of Euclidean geometry, which the circle and the triangle represent the largest part of it, and we will note that there are many problems that connect them.

Definition1: - A circle is a set of points in a plane that are equal distances from a fixed point, equal dimensions are called radii of a circle, and a fixed point is called the center of a circle. - The segment connecting the center of the circle and the circle is called a radius. - The radii of a circle are identical.	
Definition 2: - The segment connecting two points on a circle is called a chord of a circle. - If any chord passes through the center of the circle, it is called the diameter of the circle. - The secant that cuts a circle is the line that contains the chord of this circle.	
Definition 3: Identical circles are circles that have the same length of radius.	
Definition4: Concentric circles are circles that have the same center.	

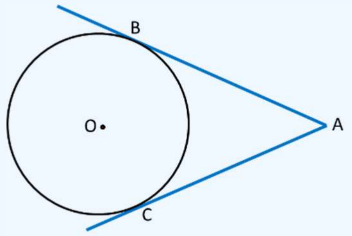
Definition5: A polygon created within a circle is a polygon whose vertices are located on the circle, and a circle is called the Circumcircle of the polygon.	
Definition 6: A tangent line of a circle is a line that touches a circle in the same plane at a single point called point of contact.	
Theory 1: The radius is perpendicular to the tangent at the point of contact.	
Theory 2: The two tangents drawn from a single point outside the circle are congruent.	
Definition 7: The common tangent of two circles in the same plane is called an inner common tangent if the line of the two centers is crossed, and it is an outer common tangent if	
Definition 8: tangent circles in the same plane are the two circles that touch the same line at the same point. In two tangent circles, their centers and contact point are on	
Definition 9: When the sides of a polygon touch a circle, this polygon is called the tangential polygon of the circle, and the circle is called the incircle of the polygon.	

Exercises:

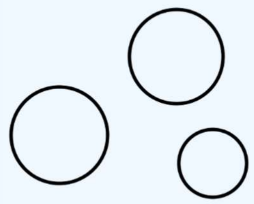
(1) In the figure, two tangents $\overrightarrow{AB}, \overrightarrow{AC}$ of the circle O,

prove that:

a) $\angle ABO = \angle ACO$
b) $AB = AC$
c) $\angle AOB = \angle AOC$
d) $\angle BAO = \angle CAO$
e) If it is a point of intersection of AO with BC is D, prove that, $BD \perp AD$

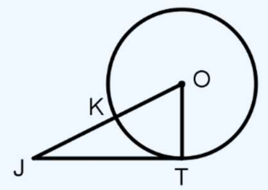


(2) On the adjacent diagram: Find the number of common tangents to each pair of the three circles



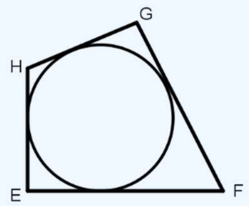
(3) On the next figure:

a) If $JO = 13$ $OT = 5$, then $JT =$ ____________
b) If $m\angle OJT = 30°, JO = 20$, then $JT =$ ________
c) If $JK = 9$, $KO = 8$, then $JT =$ ________



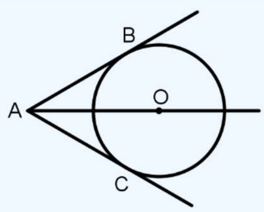
(4) On the next figure:

If HGFE a circumscribed polygon of the circle, discover and then prove the relation between $HG + EF$, $HE + GF$.



(5) On the next figure:

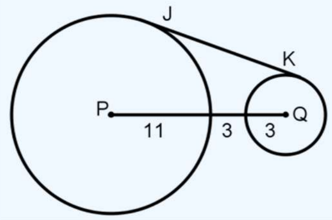
If $\overrightarrow{AB}, \overrightarrow{AC}$ tangents of the circle O, discover and then prove the relation between $\overrightarrow{AO}$, $\angle BAC$.

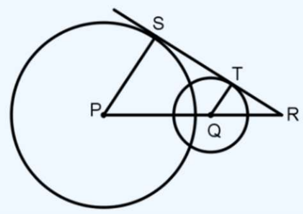
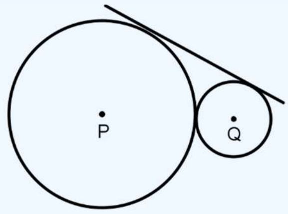
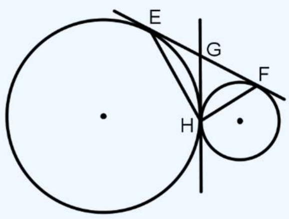
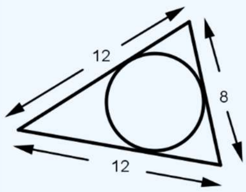
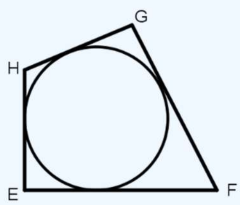


(6) On the next figure:

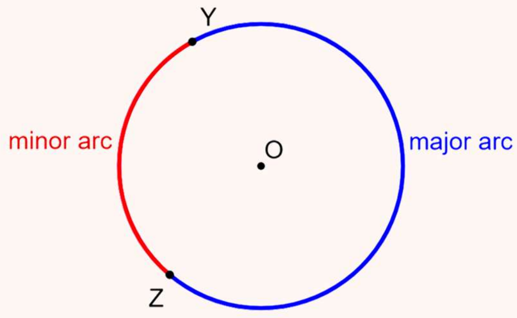
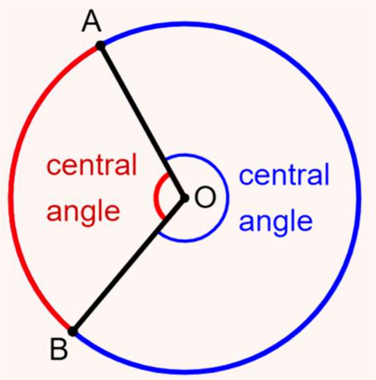
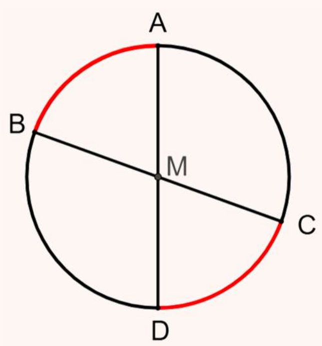
If JK is an outer tangent of the two circles P, Q, find length of JK

(Hence: What kind of quadruple JPQK)



(7)	On the next figure: If the two circles P, Q are tangential to the outer tangent SR, $QT = 6 , TR = 8 , PR = 30$ then find the length of PS , PQ , ST	
(8)	On the next figure: If the radii of the circles P, Q are $6 , 2$ respectively, find the length of the outer common tangent of the two circles.	
(9)	On the next figure: Two tangent circles, $\overline{EF}$ is common outer tangent, $\overline{GH}$ is common inner tangent, find the measure of $\angle EHF$, with proof	
(10)	On the next figure: Find the length of the radius of the inner circle of the .triangle	
(11)	On the adjacent figure, an incircle is drawn in the quadrilateral $EFGH$ if $\overline{EF} = 16 , \overline{FG} = 15 , \overline{GH} = 12$ Find length of $\overline{HE}$	

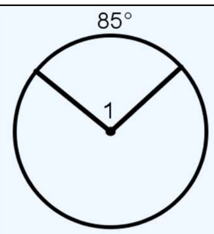
Arcs and central angles in the circle

Definition10: - The arc is a part of the circle, so if we have two points Y, Z on the circle O, they are always the end points of two arcs in the circle, one of them called the minor arc and the other is called the major arc - if $\overline{YZ}$ is the diameter of the circle, then the two arcs each represent half of the circle.	
Definition11: - The central angle is the angle that is headed by the center of the circle, and its sides are radii in the same circle. It is confined to an arc in the circle called the arc of the central angle. - The measurement of the arc in a circle is equal to the measurement of its central angle , for example: if the $m\angle AOB = 125°$ then $m\widehat{AB} = 125°$	
Definition12: Congruent Arcs in one or two congruent circles have the same measurement	
Theory 3: In the same circle or in congruent circles, the arcs are congruent if and only if the central angles that confine them are congruent , for example: $\angle CMD \cong \angle AMB \Leftrightarrow \widehat{AB} \cong \widehat{CD}$	

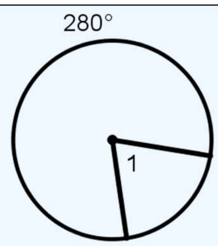
Exercises:

In Exercises 1–6, look for the measurement of the center angle $m\angle 1$

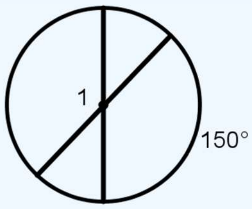
(1)



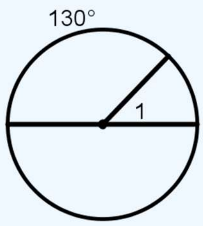
(2)



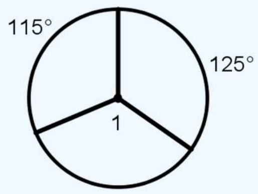
(3)



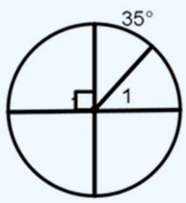
(4)



(5)



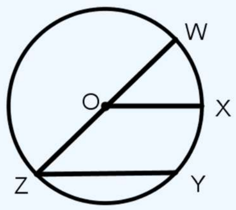
(6)



(7) On the adjacent shape:

$\overline{WZ}$ is diameter in the circle O, $\overline{OX} \parallel \overline{ZY}$.

Prove that $\overset{\frown}{WX} \cong \overset{\frown}{XY}$



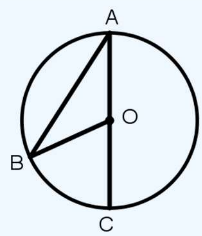
(8) On the Adjacent Shape: $\overline{AC}$ is Diameter in the Circle O

a) If $m\angle A = 35° \Rightarrow m\angle B =$ ________

، $m\angle BOC =$ ________ $m\,\overset{\frown}{BC} =$ ________

(b If $m\angle A = n \Rightarrow m\,\overset{\frown}{BC} =$ ________

(c If $m\,\overset{\frown}{BC} = 6k \Rightarrow m\angle A =$ ________

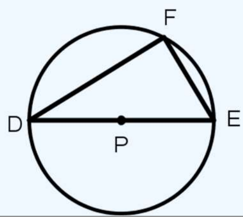


(9) On the next figure:

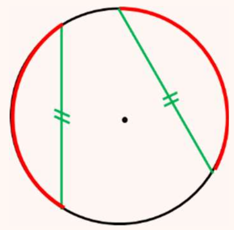
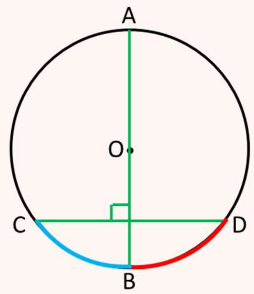
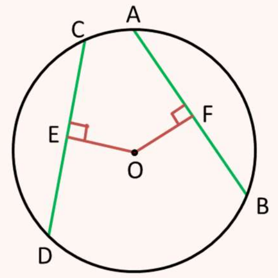
$\overline{DE}$ is diameter in circle P, $m\,\overset{\frown}{EF} = n$,

a) Find $m\angle DEF$

b) Find $m\angle DFE$



Arcs and chords

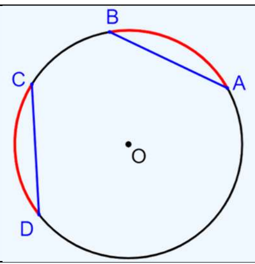
Theory 4: in the same circle or congruent circles, 1) Congruent chords enclose congruent arches. 2) Congruent arches enclose congruent chords	
Theory 5: in the circle, the perpendicular diameter to the chord halves the arc that this chord is confined to. For example: in the adjacent figure, the diameter $\overline{AB}$ divides the chord $\overline{CD}$ in half and the arc $\widehat{CD}$ splits in half $\widehat{CB} \cong \widehat{BD}$	
Theory 6: in the same circle or congruent circles, Congruent chords are equal distances from the center of the circle (or centers of circles) Chords that have equal distances from the center of the circle are congruent For example, $\overline{AB} \cong \overline{CD} \Leftrightarrow \overline{OF} \cong \overline{OE}$	

Exercises:

(1) On the adjacent figure, prove that:

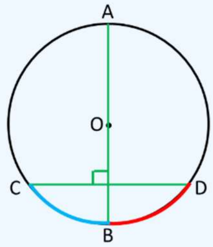
a) If $\overline{AB} \cong \overline{CD}$ *then* $\widehat{AB} \cong \widehat{CD}$

b) If $\widehat{AB} \cong \widehat{CD}$ *then* $\overline{AB} \cong \overline{CD}$



(2) In the adjacent figure, $\overline{AB}$ is diameter, $\overline{AB} \perp \overline{CD}$

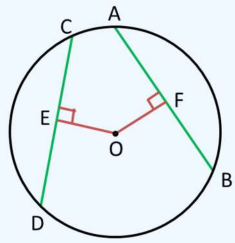
Prove that $\widehat{CB} \cong \widehat{DB}$



(3) In the adjacent figure, prove that:

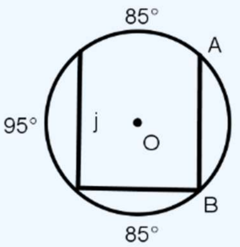
a) If $AB = CD$ then $OF = OE$

b) if $OF = OE$ then $AB = CD$



(4) In the adjacent figure,

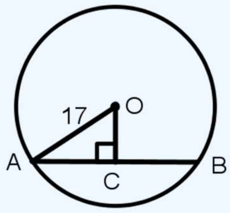
$\overline{AB} =$ ______



(5) In the adjacent figure,

if $\overline{AB} = 30$

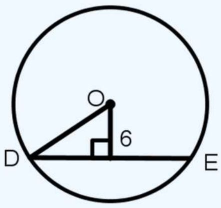
then $\overline{OC} =$ ______



(6) In the adjacent figure,

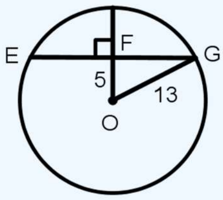
if $\overline{DE} = 16$

then $\overline{OD} =$ ______



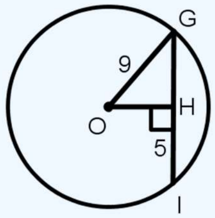
(7) In the adjacent figure,

$\overline{EG} =$ ______



(8) In the adjacent figure,

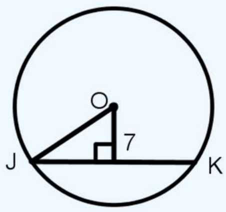
$\overline{OH} =$ ______



(9) In the adjacent figure,

if $\overline{JK} = 14$

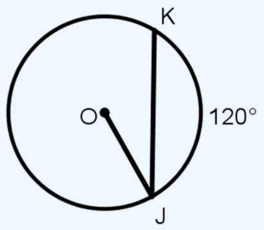
then $\overline{OJ} =$ ______



(10) In the adjacent figure,

if $\overline{OJ} = 12$

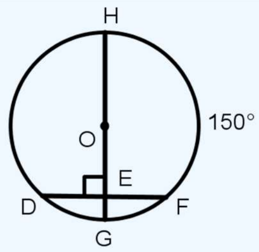
then $\overline{JK} =$ ______



(11) In the adjacent figure,

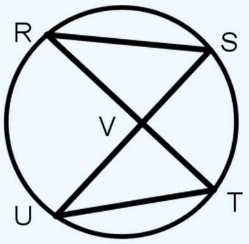
if $\overline{OE} = 8\sqrt{3}$

then $\overline{HG} =$ ______



(12) On the adjacent shape, $\overline{RS} \cong \overline{UT}, \angle R \cong \angle U$

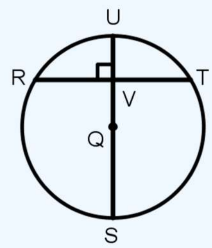
Prove that $\overline{VS} \cong \overline{VT}$, $\overline{RV} \cong \overline{UV}$.



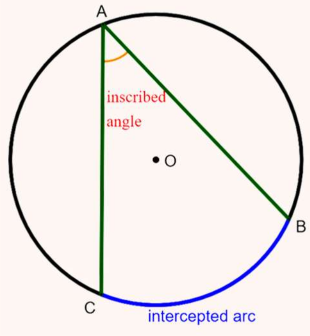
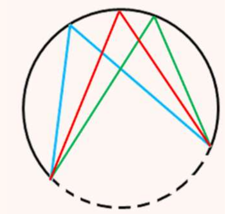
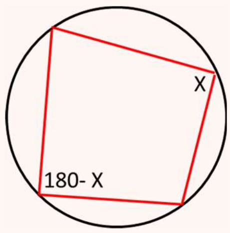
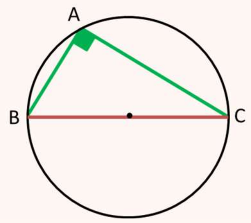
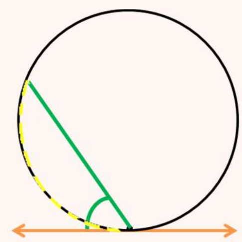
(13) On the adjacent shape, in the circle Q, the diameter $\overline{\mathrm{US}}$ is perpendicular to the chord $\overline{\mathrm{RT}}$,

If $\overline{\mathrm{RT}} = 16$, $\overline{\mathrm{QS}} = 10$

Find length of $\overline{UV}$



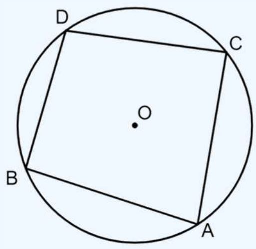
Inscribed angles and tangential angles

<table>
<tr><td>Definition13:
The inscribed angle is the angle whose head is on the circumference of the circle, and its sides are two chords</td><td rowspan="2">
</td></tr>
<tr><td>Theory 7:
The measurement of the inscribed angle is equal to half of the measurement of the intercepted arc.
For example: in the adjacent figure, $m\angle A = \frac{1}{2} m\,\widehat{BC}$</td></tr>
<tr><td>Result 1:
The measurements of the inscribed angles that confine the same arc in the same circle are equal</td><td></td></tr>
<tr><td>Result 2:
The two opposite angles in the cyclic quadrilateral are complementary</td><td>
</td></tr>
<tr><td>Result 3:
inscribed angle drawn in a semicircle is right angle</td><td>
</td></tr>
<tr><td>Theory 8:
The measurement of the angle between a chord and a tangent at a point of contact is equal to half the measurement of the arc confined between this chord and tangent</td><td></td></tr>
</table>

Exercises:

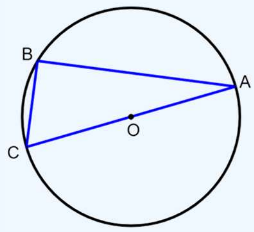
(1) On the adjacent figure, prove that:

$m\angle A + m\angle D = 180°$



(2) In the adjacent figure, $\overline{AC}$ is diameter,

Prove that $m\angle B = 90°$

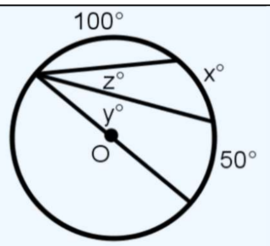


(3) In the adjacent figure,

$x =$ ______

$y =$ ______

$z =$ ______

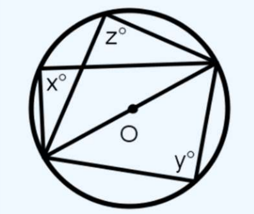


(4) In the adjacent figure,

$x =$ ______

$y =$ ______

$z =$ ______

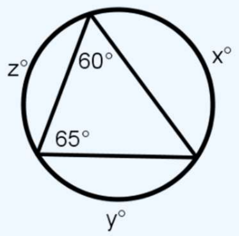


(5) In the adjacent figure,

$x =$ ______

$y =$ ______

$z =$ ______

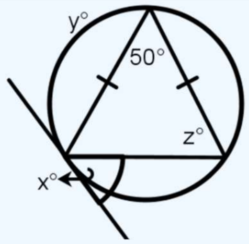


(6) In the adjacent figure,

$x =$ ______

$y =$ ______

$z =$ ______

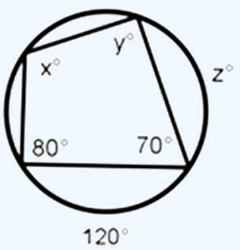


(7) In the adjacent figure,

$x =$ ______

$y =$ ______

$z =$ ______

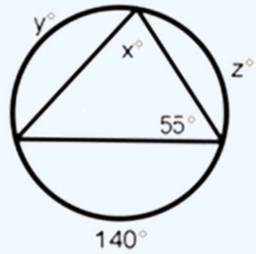


(8) In the adjacent figure,

$x =$ ______

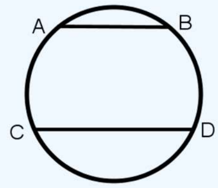
$y =$ ______

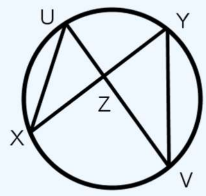
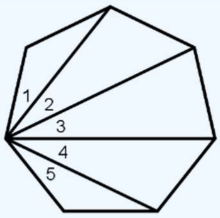
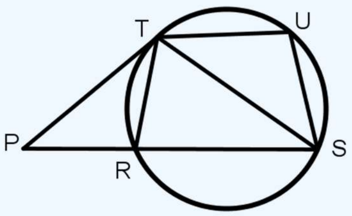
$z =$ ______



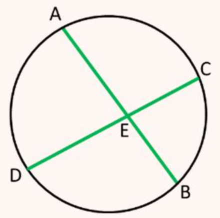
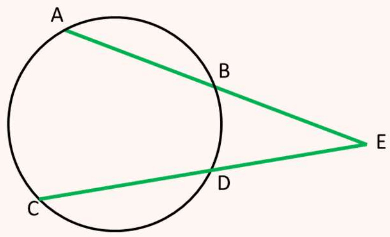
(9) On the adjacent figure, prove that if two chords in a circle are parallel, they will confine two congruent arcs

$\overline{AB} \parallel \overline{CD} \Rightarrow \overset{\frown}{AC} \cong \overset{\frown}{BD}$



(10)	On the adjacent figure, *a)* prove that $\Delta UXZ \sim \Delta YVZ$ b) prove that $ZY \cdot ZX = ZU \cdot ZV$	
(11)	On the adjacent figure, find the numbered angle measurements if the shape represents a regular heptagon	
(12)	On the adjacent figure, $\overline{PT}$ is tangent to the circle, $\overline{TU} \parallel \overline{PS}$. find three similar triangles and prove the similarity	

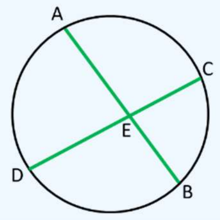
Angles between two secant lines

Theory 9: If two chords intersect within a circle, the angle between them is equal to half of the sum of the intercepted arcs. For example: in the adjacent figure, $\angle AEC = \frac{1}{2}[\widehat{AC} + \widehat{BD}]$	
Theory 10: If two secant lines intersect outside a circle, the angle between them is equal to half of the difference between the two measures of the opposite arcs. For example: in the adjacent figure, $\angle AEC = \frac{1}{2}[\widehat{AC} - \widehat{BD}]$	

Exercises:

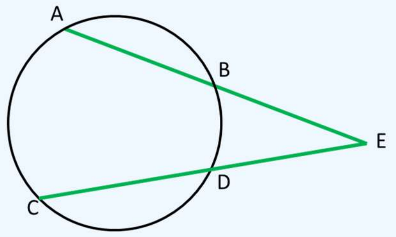
(1) On the adjacent figure, prove that:

$\angle AEC = \frac{1}{2}[\widehat{AC} + \widehat{BD}]$

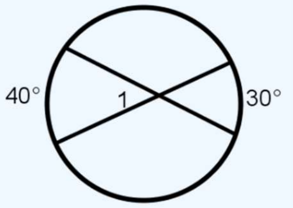


(2) In the adjacent figure, prove that:

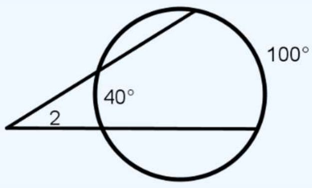
$\angle AEC = \frac{1}{2}[\widehat{AC} - \widehat{BD}]$



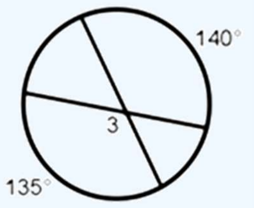
(3) In the adjacent figure, $m\angle 1 =$ ______



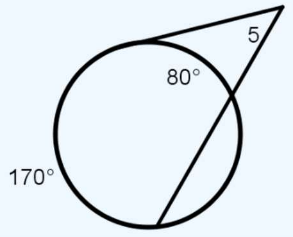
(4) In the adjacent figure, $m\angle 2 =$ ______



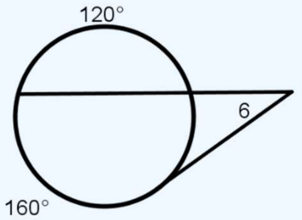
(5) In the adjacent figure, $m\angle 3 =$ ______



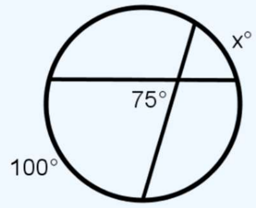
(6) In the adjacent figure, $m\angle 5 =$ ______



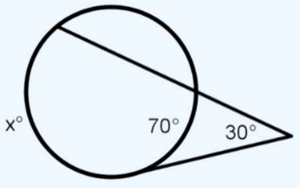
(7) In the adjacent figure, $m\angle 6 =$ ______



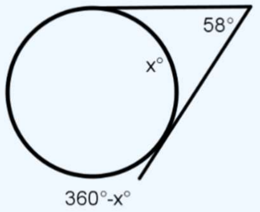
(8) In the adjacent figure, $x =$ ______



(9) In the adjacent figure, $x =$ _____



(10) In the adjacent figure, $x =$ ______



(11) On the adjacent shape, $\overleftrightarrow{BZ}$ is tangent to the circle O, $\overline{AC}$ is dimeter, $m\,\widehat{DE} = 20°, m\,\widehat{BC} = 90°, m\,\widehat{CD} = 30°$, find:

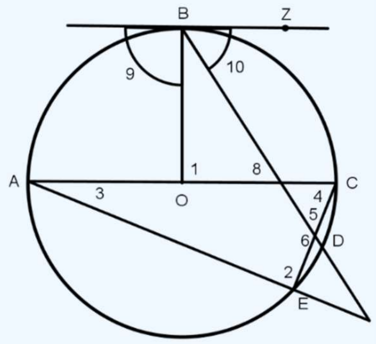
$m\angle 1 =$ ______ , $m\angle 2 =$ ________

$m\angle 3 =$ ______ , $m\angle 4 =$ ________

$m\angle 5 =$ ______ , $m\angle 6 =$ ________

$m\angle 8 =$ ______ , $m\angle 9 =$ ________

$m\angle 10 =$ ________



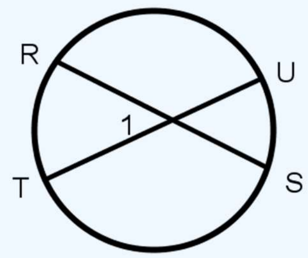
(12) On the adjacent shape,

a) If $m\,\widehat{RT} = 80°, m\,\widehat{US} = 40°$ then $m\angle 1 =$ _____

b) If $m\,\widehat{RU} = 130°, m\,\widehat{TS} = 100°$ then $m\angle 1 =$ _____

c) If $m\,\widehat{RT} = 70°, m\angle 1 = 50°$ then $m\,\widehat{US} =$ ______

d) If $m\,\widehat{US} = 36°, m\angle 1 = 52°$ then $m\,\widehat{RT} =$ ______

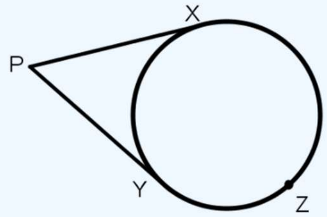


(13) On the adjacent shape, Two tangents $\overrightarrow{PX}, \overrightarrow{PY}$

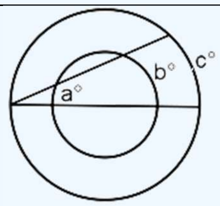
a) if $m\,\widehat{XZY} = 250°$ then $m\angle P =$ __________

b) if $m\,\widehat{XY} = 90°$ then $m\angle P =$ __________

c) if $m\angle P = 85°$ then $m\,\widehat{XY} =$ __________

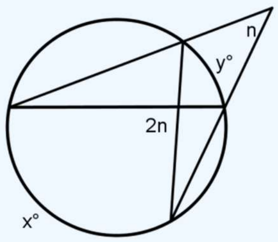


(14) On the adjacent figure, write an equation that contains a, b, c



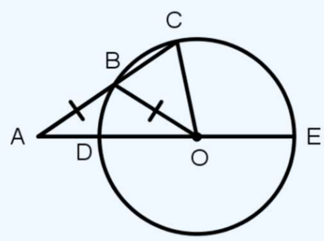
(15) On the next diagram,

find the ratio $x : y$



(16) On the adjacent shape, $\overline{AC}, \overline{AE}$ two intersections of the circle O, $\overline{AB} \cong \overline{OB}$

Discover and prove the relations between $m\,\widehat{CE}$, $m\,\widehat{DB}$



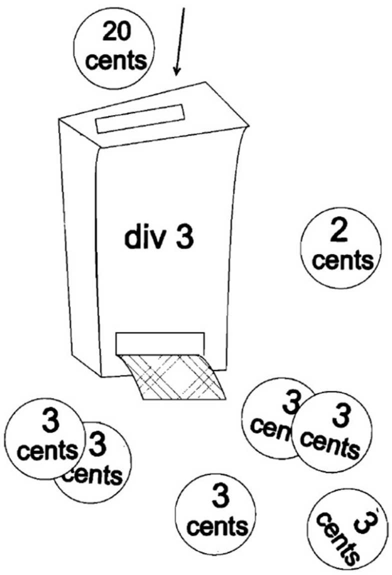
Third Unit: Number Theory

Theory of Remainders

Introduction: When we divide an integer a by an integer b, in most cases there will be a remainder (for example: the remainder of 12 divided by 5 is 2). In this chapter, we will focus our attention on the remainders. There are many real-life and scientific applications that rely on the remainders; for instance, cybersecurity—which enables us to encrypt messages, secure funds in banks, and set passwords—relies primarily on the theory of remainders that we will study in this unit. Moreover, we will look at some exercises and ideas are included to reinforce the understanding of this topic.

Motivating example: Suppose you are in a country where the currencies have set values, and you want to buy juice for 3 cents from a vending machine. You have a 15-cent coin in your pocket, but you do not have any of the 3-cent coins you need to buy the juice. Fortunately, you see a change machine that dispenses any number of 3-cent coins. Clearly, you get five 3-cent coins in exchange for the 15-cent coin you have. What if you had a 20-cent coin? Of course, you would get six 3-cent coins plus two-cent change. Therefore, we have $20 = 6 \times 3 + 2$ (see the following figure). This shows the process of dividing 20 by 3 with a remainder to get a remainder of 2.

How does our change machine work? It gives number 3-cent coins. After that, it gives you a coin for the remainder, which could be 0, 1-coin, or 2-coin.



Clearly, the remainder is zero if and only if the original number (the value of the coin you put into the machine) is divisible by 3. Similarly, we can imagine a machine that gives m-cent coins; it will change for you a number of m-cent coins and a remainder ranging in value $m-1$ from 0 to cents. This machine would represent the process of division by m with a remainder. Now we present a more precise definition: when dividing a natural number N by a natural number m , we can write it as follows:

$$N = k \cdot m + r$$

where $r \leq m - 1$. We will call the number r the remainder when we divide N by m. As an example, if $N = 92$ and $m = 17$, then $r = 7$ because:

$$92 = 5 \times 17 + 7$$

Example 1:

Find the remainder of dividing 277 by 32.

Solution: By applying the formula above, we can write 277 as follows:

$$277 = 8 \times 32 + 21$$

Therefore, the remainder is 21.

Now we can discuss the following **problem:**

A person put twenty-two 50-cent coins and forty-four 10-cent coins into the change machine. What is the remainder after they receive 3-cent coins?

This is easy. It is enough to find the remainder when $x = 22 \cdot 50 + 44 \cdot 10$.

What is noteworthy is that we do not have to calculate all the products. Suppose we replace all numbers with the remainder when dividing by 3. The number will become $x = 1 \cdot 2 + 2 \cdot 1$. This is the number 4, which has a remainder of 1 when divided by 3. We claim that the remainder when x is divided by 3 is also 1. The reason is as follows:

Theory of Remainders:

The remainder of the sum (or product) of two **natural numbers** divided by 3 is equal to the remainder of the sum (or product) of the two **remainders** of those numbers when divided by 3.

We will prove the multiplication case and leave the addition case as an exercise:

If:

$$N_1 = k_1 \cdot 3 + r_1, N_2 = k_2 \cdot 3 + r_2.$$

Then,

$$N_1 \cdot N_2 = (3 \cdot k_1 + r_1)(3 \cdot k_2 + r_2) = 3^2 \cdot k_1 k_2 + 3 \cdot k_1 r_2 + 3 \cdot k_2 r_1 + r_1 r_2$$
$$= 3(3k_1 k_2 + k_1 r_2 + k_2 r_1) + r_1 r_2.$$

Consequently, when changing $N_1 \cdot N_2$ cents, the machine will give a number of 3-cent coins equal to $(3k_1k_2 + k_1r_2 + k_2r_1)$, and r_1r_2 will remain. Therefore, the remainder of putting in $N_1 \cdot N_2$ is the same as the remainder of r_1r_2 .

Important note: the divisor 3 can be changed to any other natural number (the proof idea does not differ from the case of 3).

Example 2:

Find the remainder when dividing:

1. the number $2023 \cdot 2024 \cdot 2025 + 2026^3$ by 7.
2. the number 9^{100} by 8.

Solution:

1. The remainder is equal to the same remainder of $0 \cdot 1 \cdot 2 + 3^3$ divided by 7, which equals the same division of $0 + 27$ by 7, which equals 6.
2. The remainder is equal to the same remainder of 1^{100} divided by 8, which equals the same division of 1 by 8, which equals 1.

Example 3:

Prove that:

$n^3 + 2n$ is divisible by 3 for any natural number n.

Solution:

When dividing the number n by 3, the remainder is either 0, 1, or 2; thus, we consider three cases.

- If the remainder of dividing the number n by 3 is 0, then both n^3 and $2n$ will be divisible by 3, and consequently $n^3 + 2n$ is divisible by 3.

- If the remainder of dividing the number n by 3 is 1, then n^3 has the remainder 1, $2n$ has the remainder 2, and $1 + 2$ is divisible by 3.

- If the remainder of dividing the number n by 3 is 2, then n^3 has the remainder 2, $2n$ has the remainder 1, and $2 + 1$ is divisible by 3.

Thus, in all cases, the requirement is proven.

Exercises:

1) Find the remainder when 5 is divided by 7.

2) Given two natural numbers $m < n$. Find the remainder when m is divided by n.

3) Find the remainder when dividing:

   a) The number $2015 \cdot 2016 \cdot 2017 + 2018^3$ by 9.

   b) The number 8^{100} by 7

4) If the remainder of dividing N_1 by 3 equals r_1 and the remainder of dividing N_2 by 3 equals r_2, prove that the remainder of dividing $N_1 + N_2$ by 3 equals the remainder of dividing $r_1 + r_2$.

Prove that:

5) $n^5 + 4n$ is divisible by 5 for any natural number n.

6) $n^2 + 1$ is NOT divisible by 3 for any natural number n.

7) $n^3 + 2$ is NOT divisible by 9 for any natural number n.

8) $n^3 - n$ is divisible by 24 for any odd natural number n.

9) $p^2 - 1$ is divisible by 24 for any prime number $p > 3$.

10) $p^2 - q^2$ is divisible by 24 for any prime numbers $p, q > 3$.

11) (Challenge) Given three natural numbers x, y, z that satisfy the equation

$$x^2 + y^2 = z^2.$$

Prove that at least one of them is divisible by 3.

12) (Challenge) Given two natural numbers a, b such that $a^2 + b^2$ is divisible by 21. Prove that $a^2 + b^2$ is divisible by 441.

13) (Challenge) Given three natural numbers a, b, c that satisfy that $a + b + c$ is divisible by 6. Prove that $a^3 + b^3 + c^3$ is also divisible by 6.

14) (Challenge) Given three prime numbers p, q, r that satisfy that:

$$q = p + d, r = p + 2d$$

For some natural number d. Prove that d is divisible by 6.

Fourth Unit: Combinatorics

Mixed Exercises on Counting Principles

(1) We have three rooms: the first holds 1 person, the second holds 2 people, and the third holds 4 people. If we want to assign 7 students to these three rooms, how many different assignments are possible?

(2) How many 10-digit positive integers contain at least two identical digits?

(3) How many strings can be formed using only the letters A and B, containing exactly five A's and at most three B's?

(4) Is the number of 7-digit positive integers that do not contain the digit 1 equal to half the total number of 7-digit positive integers?

(5) How many 10-digit positive integers have the property that the sum of their digits is even?

(6) In how many ways can we place 4 identical rooks (same color) on a chessboard such that no two rooks attack each other?

(7) لWe have 6 distinct books to arrange on a shelf, with the condition that two specific books (for example, A and B) must always be adjacent. How many arrangements are possible?

(8) How many ways are there to choose a 3-person committee from 10 people, given that Jawad must be a member of the committee?

Ascending and Descending Order

In this lesson, we learn how to count numbers whose digits are arranged in **ascending** or **descending** order. We will see that this ordering affects the total number of possible outcomes. These ideas will be applied through examples and various exercises.

Example 1:

In how many ways can a 4-digit number be formed using the digits $\{1, 2, 3, 4, 5, 6\}$ such that its digits are arranged in ascending order?

Solution:

We have the digits: $\{1, 2, 3, 4, 5, 6\}$

We want to form a **4-digit number** such that the digits are:

- chosen from this set
- arranged in ascending order

Important note: Ascending or descending order means that **no digit can be repeated**, since the digits must be distinct.

When the digits are **arranged in ascending order**, this means that:

- We choose 4 distinct digits out of the 6 available digits.
- **The order is then automatically determined** (from smallest to largest).

Therefore, the problem reduces to:

In how many ways can we choose 4 digits from 6 without regard to order?

This is exactly the definition of **combinations**.

$$\binom{6}{4} = \frac{6!}{4!\ 2!} = \frac{6 \times 5}{2 \times 1} = 15$$

What if the set contains the digit zero?

What difference will appear in the solution?

We illustrate this with the following example:

Example 2:

In how many ways can a 4-digit number be formed using the digits $\{0,1,2,3,4,5,6\}$ such that its digits are arranged in **ascending order**?

Solution:

The set is $\{0,1,2,3,4,5,6\}$

We want to form a **4-digit number** whose digits are arranged in **ascending order.**

Important note:

Saying "a **4-digit number**" means that the number must indeed have **four digits**, so:

- The first digit **cannot be zero.**

And since the digits are **arranged in ascending order**:

The first digit represents the smallest digit in the number.

Therefore, zero cannot be used, because if it were used, it would appear in the first position, making the number not a four-digit number. Hence, we exclude zero, and the set becomes: $\{1,2,3,4,5,6\}$

At that point:

- We choose only 4 digits out of the 6,
- and their ascending order is automatically determined.

Therefore, we use combinations, just as in the previous example. $\binom{6}{4}$

Conclusion:

From the above, we conclude that when the digits are arranged in **ascending order**, the presence of zero affects the solution. This is because zero is the smallest digit and would occupy the first position, which prevents the formation of a number with the required number of digits. Hence, zero must be excluded when making the selection.

On the other hand, when the digits are arranged in **descending order**, the method of solution remains the same, and the presence or absence of zero does not affect the result. This is because the first position is occupied by the largest digit, and zero can never appear there.

Exercises:

(1) How many numbers from $100, 101, \ldots, 999$ have their three digits arranged in ascending or descending order from left to right?

(2) In how many ways can a 5-digit number be formed using digits from $\{1, 2, 3, 4, 5, 6, 7, 8\}$ such that its digits are:

(a) in ascending order?

(b) in descending order?

(3) In how many ways can a 4-digit number be formed using digits from $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ such that its digits are:

(a) in ascending order?

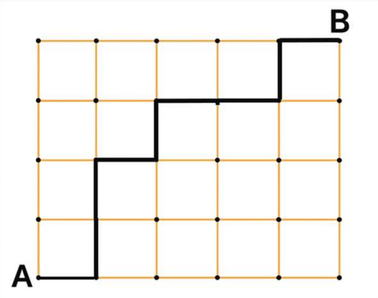
(b) in descending order?

(4) Mohammed has ten cards numbered from 1 to 10. He plays a game with his brother Salem, where Salem draws three cards in a row without replacement. Salem wins if the three cards he draws are in ascending order; otherwise, Mohammed wins. Who has the higher chance of winning, Salem or Mohammed? And what is each player's winning probability (percentage)?

Counting Paths on a Grid

Example:

We have a grid of size 4×5 How many different paths are there from point A to point B if you are only allowed to move up and to the right?



Solution:

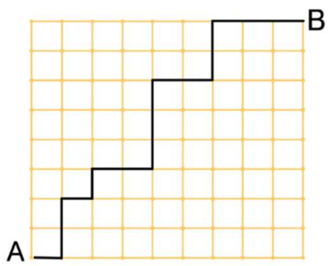
Let U represent an upward move and R represent a move to the right.

To move from point A to point B using only up and right moves, we need 4 upward moves and 5 right moves, i.e., a sequence like: $UUUURRRRR$ So the total number of moves is: $4+5=9$

The number of different paths equals the number of distinct arrangements of these moves: $\frac{9!}{4!\times 5!}$

This can also be found by counting how many ways we can choose 4 positions out of 9 to place the U moves (the remaining positions will be R moves): $\binom{9}{5}=\binom{9}{4}=126$

And in general, for an $m \times n$ grid as shown in the figure:

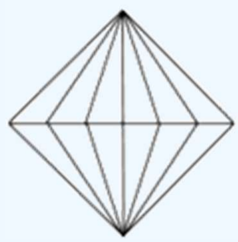


The number of shortest paths from A to B on an $m \times n$ grid is equal to: $\binom{m+n}{m}$

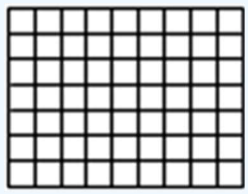
Exercises:

(1) A frog starts at the origin 0 on the number line. Each time it jumps one step, either forward (+1) or backward (−1). After 13 jumps, in how many different ways can it end up at position −1?

(2) How many triangles are there in the following figure?



(3) How many rectangles are there in the following figure?



(4) In a certain city, a license plate consists (from left to right) of two letters in alphabetical order, followed by two digits in increasing order (for example, $RE64$). If the letter O is not allowed and the digit 0 is not used, how many different license plates are possible?

(5) On a refrigerator door, there are 9 magnetic tiles, each labeled with one of the letters from the word $MATHCOUNT$. We select two vowels and three consonants from these tiles and place them into a bag. Assuming that the two T letters are indistinguishable, how many different possible collections of letters can be in the bag?

Stars and Bars

The stars and bars method is used in counting problems that require finding the number of ways to distribute n identical items among k elements (or groups). We start with a simple example to illustrate the idea.

In how many different ways can we distribute n identical objects into k distinct boxes? We represent the n objects by stars and use $k-1$ bars to divide them, in order to obtain k boxes:

$$\underbrace{**\ldots*}_{n times}\ \underbrace{||\ldots|}_{k-1 times}$$

The required number of ways is equal to the number of ways to arrange the stars and bars, which is: $\binom{n+k-1}{k-1}$

When using the stars and bars method to solve a problem, we must carefully verify that all the given conditions of the problem are satisfied.

Example:

Abdullah has 6 riyals and wants to distribute them among four of his brothers: Mohammed, Moaz, Samir, and Jaber. In how many different ways can he do this?

Solution:

We have 6 stars (*) and 3 bars (|), and there are no restrictions on the distribution.

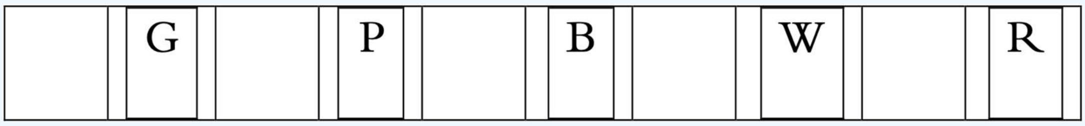
Therefore, the number of ways is:

$$\binom{6+4-1}{4-1} = \binom{9}{3} = \frac{9\cdot 8\cdot 7}{3!} = 84$$

Exercises:

(1) Ten identical boxes are to be distributed among 3 rooms, such that each room receives at least one box, and all boxes must be placed in the rooms. How many different ways can this be done?

(2) There are ten parking spaces in a single row, and five cars of different colors are to be parked, under the condition that no two cars are parked next to each other, as shown in the figure.



In how many ways can the cars be parked under this condition?

(3) How many non-negative integer solutions are there to the equation: $x_1 + x_2 + x_3 + x_4 = 80$?

(4) How many positive integer solutions are there to the equation: $x_1 + x_2 + x_3 + x_4 = 80$

(5) In a certain parking lot, there are 16 parking spaces, and Sajjad's car needs two adjacent spaces to park. If 12 small cars arrive before Sajjad, what is the probability that Sajjad will be able to park his car?

(6) There are nine seats in a row, with 6 students and 3 teachers to be seated. If the teachers arrive first and it is required that each teacher sits between two students, in how many ways can everyone be seated?

Challenge Questions

(1) From a soccer team consisting of 12 players, the management wants to choose a team of 4 players, such that Saad and Saeed cannot both be among the selected four at the same time. In how many ways can this be done?

(2) How many even numbers consisting of 6 digits have their digits arranged in descending order from left to right?

(3) During the holidays, Farah wants to distribute 15 pieces of candy among 3 children, such that each child receives at least one piece, and the first child receives at most 7 pieces. In how many ways can this be done?

(4) In how many ways can 8 identical rooks (all of the same color) be placed on a chessboard such that no two rooks attack each other?

(5) In how many ways can a group consisting of 50 students and 50 teachers have dinner in a restaurant, under the condition that each table seats exactly one student and one teacher, given that the tables are distinct and the seating positions are distinct?

(6) Eight friends are sitting around a circular table. In how many ways can they be seated such that two specific people always sit next to each other, and a third specific person does not sit next to either of them?

Solutions

Algebra Solutions

1 -Quadratic Equation:

Exercises:

(1)

$(\boldsymbol{a})\ \boldsymbol{x^2 - 12x - 540 = 0}$

We are looking for two numbers whose product is -540 and whose sum is -12

The two numbers are 18 and -30

$\Longrightarrow x^2 - 12x - 540 = (x-30)(x+18) = 0$

$\Longrightarrow x = 30\ or\ x = -18$

$(\boldsymbol{b})\ \boldsymbol{3x^2 = 10x + 24}$

$\Longrightarrow 3x^2 - 10x - 24 = 0$ "Move all terms to one side of the equation"

$a = 3, b = -10, c = -24 \Longrightarrow \Delta = b^2 - 4ac = (-10)^2 - 4(3)(-24) = 100 + 288 = 388$

$$\therefore x = \frac{10 \pm \sqrt{388}}{6} = \frac{5 \pm \sqrt{97}}{3}$$

$$\Longrightarrow x = \frac{5 + \sqrt{97}}{3} \quad or \quad x = \frac{5 - \sqrt{97}}{3}$$

$(\boldsymbol{c})\ \boldsymbol{(x^4 - 11x^3 + 24x^2) - (4x^2 - 44x + 96) = 0}$

$\Longrightarrow x^2(x^2 - 11x + 24) - 4(x^2 - 11x + 24) = 0$

$\Longrightarrow (x^2 - 11x + 24)(x^2 - 4) = 0$

$\Longrightarrow (x-8)(x-3)(x-2)(x+2) = 0$

$\Longrightarrow x = 8\ or\ 3\ or\ 2\ or -2$

(2)

$ax^2 - 5x + 9 = 0$

The equation has exactly one solution if the discriminant equals zero ($\Delta = 0$) that is:

$\Delta = b^2 - 4ac = (-5)^2 - 4(a)(9) = 0$

$\Longrightarrow 25 - 36a = 0$

$$\Longrightarrow a = \frac{25}{36}$$

(3)

$(x^2 - x - 1)^{x+2} = 1$

Case 1: The exponent is equal to zero (provided the base is non-zero)

$x + 2 = 0 \Longrightarrow x = -2$

When $x = -2$ the base is non-zero.

Case 2: The base is equal to 1.

$x^2 - x - 1 = 1 \Longrightarrow x^2 - x - 2 = 0 \Longrightarrow (x-2)(x+1) = 0 \Longrightarrow x = 2 \; or \; x = -1$

Case 3: The base is equal to (-1) (provided the exponent is an even number)

$x^2 - x - 1 = -1 \Longrightarrow x^2 - x = 0 \Longrightarrow x(x-1) = 0 \Longrightarrow x = 0 \; or \; x = 1$

When $x = 0$ the exponent is $x + 2 = 0 + 2 = 2$ (an even number).

Therefore, the equation holds.

When $x = 1$ the exponent is $x + 2 = 1 + 2 = 3$ (an odd number).

Therefore, the equation does not hold.

Conclusion: The integer values of x that satisfy the equation are $\{-2, 2, -1, 0\}$ totaling four values.

(4)

Using Vieta's Formulas. Let the second root of the equation:

$$a(b-c)x^2 + b(c-a)x + c(a-b) = 0$$

be r The product of the roots is given by the relation:

$$r \cdot 1 = r = \frac{c(a-b)}{a(b-c)}$$

(5)

Since a and b are the roots of $x^2 - mx + 2 = 0$, the product of the roots is:

$$ab = 2$$

Given that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are roots of $x^2 - px + q = 0$, their product is q:

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 2 + \frac{1}{ab} = q$$

$$\Rightarrow q = 2 + 2 + \frac{1}{2} = \frac{9}{2}$$

(6)

$$\begin{cases} x^2 + xy = 39 \longrightarrow (1) \\ x - y = -33 \longrightarrow (2) \\ y + z = 13 \longrightarrow (3) \end{cases}$$

From equation (2) , we find that $y = x + 33$. Substituting this into equation(1), we obtain:

$$x^2 + x(x + 33) = 39$$
$$\Rightarrow x^2 + x^2 + 33x - 39 = 0$$
$$\Rightarrow 2x^2 + 33x - 39 = 0$$

Using the **Quadratic Formula**, where $a = 2, b = 33, c = -39$

$$\Delta = b^2 - 4ac = 33^2 - 4(2)(-39) = 1401$$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$\Rightarrow x = \frac{-33 \pm \sqrt{1401}}{2(2)}$$

$$\Rightarrow x = \frac{-33 \pm \sqrt{1401}}{4}$$

By substituting the values of x back into equations (2) and (3), we find y and z:

$$y = x + 33 \Rightarrow y = \frac{-33 \pm \sqrt{1401}}{4} + 33 = \frac{99 \pm \sqrt{1401}}{4}$$

$$z = 13 - y \Rightarrow z = 13 - \frac{99 \pm \sqrt{1401}}{4} = \frac{-47 \mp \sqrt{1401}}{4}$$

Therefore, the system has two solutions:

$$\left(\frac{-33 + \sqrt{1401}}{4}, \frac{99 + \sqrt{1401}}{4}, \frac{-47 - \sqrt{1401}}{4}\right), \left(\frac{-33 - \sqrt{1401}}{4}, \frac{99 - \sqrt{1401}}{4}, \frac{-47 + \sqrt{1401}}{4}\right)$$

(7)

Assume the equation is of the form $p(x) = ax^2 + bx + c = 0$. Since $P(0) = -1$, then:

$P(0) = a(0)^2 + b(0) + c = -1 \Longrightarrow c = -1$

Given that $P(1) = 9$, then:

$P(1) = a(1)^2 + b(1) - 1 = 9 \Longrightarrow a + b = 10 \longrightarrow (1)$

Given that $P(2) = 25$, then:

$P(2) = a(2)^2 + b(2) - 1 = 25 \Longrightarrow 4a + 2b = 26 \Longrightarrow 2a + b = 13 \longrightarrow (2)$

By subtracting equation (1) from equation (2), we obtain:

$a = 3 \Longrightarrow b = 7$

Therefore:

$P(-1) = 3(-1)^2 + 7(-1) - 1 = 3 - 7 - 1 = -5$

(8)

For the equation $x^2 + kx + 27 = 0$ to have two distinct real roots, the discriminant $\Delta > 0$

$\Delta = k^2 - 4(1)(27) = k^2 - 108 > 0$

$\Longrightarrow k^2 > 108 \Rightarrow |k| > \sqrt{108} = 6\sqrt{3}$

Hence:

$k > 6\sqrt{3} \quad or \quad k < -6\sqrt{3}$

(9)

$\frac{a+b}{a} = \frac{b}{a+b}$

$\Longrightarrow (a+b)^2 = ab$

$\Longrightarrow a^2 + 2ab + b^2 = ab$

$\Longrightarrow a^2 + ab + b^2 = 0$

Using the **Quadratic Formula** to solve for a, we find the discriminant:

$\Delta = (b)^2 - 4(1)(b^2) = -3b^2$

- If $b \neq 0$ then $\Delta < 0$. Consequently, there are no real solutions for a.
- If $b = 0$, the original equation becomes $\frac{a}{a} = \frac{0}{a} \Rightarrow 1 = 0$ This is impossible.

Therefore, $a \ and \ b$ cannot both be real numbers.

(10)

$(2 + (2 + (2 + (2 + x)^2)^2)^2)^2 = 15129$

By taking the square root of both sides (neglecting the negative result since the left-hand side is positive):

$\Longrightarrow 2 + (2 + (2 + (2 + x)^2)^2)^2 = 123$

$\Longrightarrow (2 + (2 + (2 + x)^2)^2)^2 = 121$

Taking the square root again:

$\Longrightarrow 2 + (2 + (2 + x)^2)^2 = 11$

$\Longrightarrow (2 + (2 + x)^2)^2 = 9$

Following the same procedure:

$\Longrightarrow 2 + (2 + x)^2 = 3$

$\Longrightarrow (2 + x)^2 = 1$

$\Longrightarrow 2 + x = \pm 1$

$\Longrightarrow x = -1 \ \ or - 3$

(11)

Given: $x^2 + y = 12 = y^2 + x$ Thus, $x^2 + y = y^2 + x$

$\Longrightarrow x^2 - y^2 + y - x = 0$
$\Longrightarrow (x^2 - y^2) - (x - y) = 0$
$\Longrightarrow (x - y)(x + y) - (x - y) = 0$
$\Longrightarrow (x - y)(x + y - 1) = 0$

This gives us two cases:

$\mathbf{1)}\ \boldsymbol{x - y = 0}$

$\Longrightarrow x = y$

Substituting into the first equation:

$x^2 + x = 12 \Longrightarrow x^2 + x - 12 = 0 \Longrightarrow (x + 4)(x - 3) = 0 \Longrightarrow (x, y) = (-4, -4), (3,3)$

$\mathbf{2)}\ \boldsymbol{x + y - 1 = 0}$

$\Longrightarrow y = 1 - x$

Substituting into the first equation:

$x^2 + 1 - x = 12 \Longrightarrow x^2 - x - 11 = 0$

Using the quadratic formula where $a = 1, b = -1, c = -11$

$\Delta = b^2 - 4ac = 1 - 4(1)(-11) = 45$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$\Longrightarrow x = \frac{1 \pm 3\sqrt{5}}{2}$$

$$\Longrightarrow y = 1 - \frac{1 \pm 3\sqrt{5}}{2} = \frac{1 \mp 3\sqrt{5}}{2}$$

The solution set for the system is:

$$(x, y) = (-4, -4), (3,3), \left(\frac{1 + 3\sqrt{5}}{2}, \frac{1 - 3\sqrt{5}}{2}\right), \left(\frac{1 - 3\sqrt{5}}{2}, \frac{1 + 3\sqrt{5}}{2}\right)$$

(12)

$$\begin{cases} 2x_1 = x_5^2 - 23 \\ 4x_2 = x_1^2 + 7 \\ 6x_3 = x_2^2 + 14 \\ 8x_4 = x_3^2 + 23 \\ 10x_5 = x_4^2 + 34 \end{cases}$$

Summing all equations and rearranging terms:

$$(x_1^2 - 2x_1) + (x_2^2 - 4x_2) + (x_3^2 - 6x_3) + (x_4^2 - 8x_4) + (x_5^2 - 10x_5) + 55 = 0$$

By completing the square for each expression in parentheses:

$$(x_1^2 - 2x_1 + 1) + (x_2^2 - 4x_2 + 4) + (x_3^2 - 6x_3 + 9) + (x_4^2 - 8x_4 + 16) + (x_5^2 - 10x_5 + 25) = 0$$

$$\Longrightarrow (x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2 + (x_4 - 4)^2 + (x_5 - 5)^2 = 0$$

Since the sum of squares equals zero, each individual term must be zero:

$$x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$$

2 - Other Forms of Identities:

Exercises:

(1)

$a)\ (a+b)^2-(a-b)^2=(a^2+2ab+b^2)-(a^2-2ab+b^2)$
$=a^2+2ab+b^2-a^2+2ab-b^2$
$=4ab$

$b)\ (a-b)^3+3ab(a-b)=a^3-3a^2b+3ab^2-b^3+3a^2b-3ab^2$
$=a^3-b^3$

$c)\ (a+b)^3-3ab(a+b)=a^3+3a^2b+3ab^2+b^3-3a^2b-3ab^2$
$=a^3+b^3$

$d)\ \sqrt{(a+b)^2-4ab}=\sqrt{a^2+2ab+b^2-4ab}$
$=\sqrt{a^2-2ab+b^2}$
$=\sqrt{(a-b)^2}$
$=a-b$

(2)

$a)\ a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}[2a^2+2b^2+2c^2-2ab-2bc-2ca]$
$=\frac{1}{2}[(a^2-2ab+b^2)+(a^2-2ac+c^2)+(b^2-2bc+c^2)]$
$=\frac{1}{2}[(a-b)^2+(a-c)^2+(b-c)^2]$

$b)\ (a+b+c)^3=\big((a)+(b+c)\big)^3$
$=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3$
$=a^3+3a^2b+3a^2c+3a(b^2+2bc+c^2)+(b^3+3b^2c+3bc^2+c^3)$
$=a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3$
$=a^3+b^3+c^3+3a^2b+3a^2c+3ab^2+3ac^2+3b^2c+3bc^2+6abc$
$=a^3+b^3+c^3+3(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2+2abc)$
$=a^3+b^3+c^3+3[(a^2b+ab^2)+(abc+b^2c)+(ac^2+bc^2)+(a^2c+abc)]$
$=a^3+b^3+c^3+3[ab(a+b)+bc(a+b)+c^2(a+b)+ac(a+b)]$
$=a^3+b^3+c^3+3(a+b)[ab+bc+c^2+ac]$
$=a^3+b^3+c^3+3(a+b)[b(a+c)+c(c+a)]$
$=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$

(3)

Assume $x = a^3$. Then $\sqrt[3]{x} = a$, so

$$\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}} = 3$$
$$\Longrightarrow \left(a + \frac{1}{a}\right)^3 = 3^3$$
$$\Longrightarrow a^3 + \frac{1}{a^3} + 3(a)(\frac{1}{a})(a + \frac{1}{a}) = 27$$
$$\Longrightarrow a^3 + \frac{1}{a^3} + 3(3) = 27$$
$$\Longrightarrow a^3 + \frac{1}{a^3} = x + \frac{1}{x} = 27 - 9 = 18$$
$$\Longrightarrow \left(x + \frac{1}{x}\right)^3 = 18^3$$
$$\Longrightarrow x^3 + \frac{1}{x^3} + 3(x)(\frac{1}{x})(x + \frac{1}{x}) = 5832$$
$$\Longrightarrow x^3 + \frac{1}{x^3} + 3(18) = 5832$$
$$\Longrightarrow x^3 + \frac{1}{x^3} = 5832 - 54 = 5778$$

(4)

$$\sqrt[3]{1 + \sqrt{x}} + \sqrt[3]{1 - \sqrt{x}} = 2$$
$$\Longrightarrow \left(\sqrt[3]{1 + \sqrt{x}} + \sqrt[3]{1 - \sqrt{x}}\right)^3 = 2^3$$
$$\Longrightarrow \left(\sqrt[3]{1 + \sqrt{x}}\right)^3 + \left(\sqrt[3]{1 - \sqrt{x}}\right)^3 + 3\left(\sqrt[3]{1 + \sqrt{x}}\right)\left(\sqrt[3]{1 - \sqrt{x}}\right)\left(\sqrt[3]{1 + \sqrt{x}} + \sqrt[3]{1 - \sqrt{x}}\right) = 8$$
$$\Longrightarrow 1 + \sqrt{x} + 1 - \sqrt{x} + 3\left(\sqrt[3]{(1 + \sqrt{x})(1 - \sqrt{x})}\right)(2) = 8$$
$$\Longrightarrow 2 + 6\left(\sqrt[3]{1 - x}\right) = 8$$
$$\Longrightarrow \sqrt[3]{1 - x} = 1$$
$$\Longrightarrow 1 - x = 1$$
$$\Longrightarrow x = 0$$

(5)

Assume:

$X = (a-b)(a+b-c) = a^2 + ab - ac - ab - b^2 + bc = 3$
$Y = (b-c)(b+c-a) = b^2 + bc - ab - bc - c^2 + ac = 5$
$Z = (c-a)(a+c-b) = ac + c^2 - bc - a^2 - ac + ab$
$\Longrightarrow X + Y + Z = 0$
$\Longrightarrow 3 + 5 + Z = 0$
$\Longrightarrow Z = -8$

(6)

The expression inside the radical must be greater than or equal to zero; therefore, $2 \le x \le 4$

$\sqrt{x-2} + \sqrt{4-x} = \sqrt{6-x}$
$\Longrightarrow \left(\sqrt{x-2} + \sqrt{4-x}\right)^2 = \left(\sqrt{6-x}\right)^2$
$\Longrightarrow x - 2 + 4 - x + 2\sqrt{(x-2)(4-x)} = 6 - x$
$\Longrightarrow 2 + 2\sqrt{(x-2)(4-x)} = 6 - x$
$\Longrightarrow 2\sqrt{(x-2)(4-x)} = 4 - x$
$\Longrightarrow 4(x-2)(4-x) = (4-x)^2$
$\Longrightarrow 4(x-2)(4-x) - (4-x)^2 = 0$
$\Longrightarrow (4-x)[4(x-2) - (4-x)] = 0$
$\Longrightarrow (4-x)(4x - 8 - 4 + x) = 0$
$\Longrightarrow (4-x)(5x-12) = 0$
$\Longrightarrow x = 4 \ \ or \ x = \dfrac{12}{5}$

(7)

$ab(x^2+1) = (a^2+b^2)x$
$\Longrightarrow abx^2 + ab = a^2x + b^2x$
$\Longrightarrow abx^2 - a^2x - b^2x + ab = 0$
$\Longrightarrow ax(bx-a) - b(bx-a) = 0$
$\Longrightarrow (bx-a)(ax-b) = 0$
$\Longrightarrow x = \dfrac{a}{b} \ \ or \ \ x = \dfrac{b}{a}$

(8)

Assume: $y = \sqrt{x-10}$

$$\sqrt{x-10} - \frac{6}{\sqrt{x-10}} = 5$$

$\Longrightarrow y - \frac{6}{y} = 5$

$\Longrightarrow y^2 - 6 = 5y$

$\Longrightarrow y^2 - 6 = 5y$

$\Longrightarrow y^2 - 5y - 6 = 0$

$\Longrightarrow (y-6)(y+1) = 0$

$\Longrightarrow y = 6 \;\; or \;\; y = -1$

$y = -1$ is rejected, since the principal square root is never negative.

$\Longrightarrow y = 6 \Longrightarrow \sqrt{x-10} = 6 \Longrightarrow x - 10 = 36 \Longrightarrow x = 46$

(9)

$$\frac{(2x-1)^2}{2} + \frac{(3x-1)^2}{3} + \frac{(6x-1)^2}{6} = 1$$

$\Longrightarrow 3(2x-1)^2 + 2(3x-1)^2 + (6x-1)^2 = 6$

$\Longrightarrow 3(4x^2 - 4x + 1) + 2(9x^2 - 6x + 1) + (36x^2 - 12x + 1) = 6$

$\Longrightarrow 12x^2 - 12x + 3 + 18x^2 - 12x + 2 + 36x^2 - 12x + 1 = 6$

$\Longrightarrow 66x^2 - 36x + 6 = 6$

$\Longrightarrow 66x^2 - 36x = 0$

$\Longrightarrow 6x(11x - 6) = 0$

$\Longrightarrow x = 0 \;\; or \;\; x = \frac{6}{11}$

3- Factoring:

Exercises (3-1):

(1)

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$
$$= (a+b)[(a+b)^2 - 3ab]$$
$$= (a-b)(a^2 + 2ab + b^2 - 3ab)$$
$$= (a-b)(a^2 - ab + b^2)$$

(2)

$$x^2 - (a+b)x + ab = x^2 - ax - bx + ab$$
$$= x(x-a) - b(x-a)$$
$$= (x-a)(x-b)$$

(3)

$$x^2 + (a+b)x + ab = x^2 + ax + bx + ab$$
$$= x(x+a) + b(x+a)$$
$$= (x+a)(x+b)$$

(4)

$$x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2$$
$$= (x^2)^2 + 2x^2 + 1 - x^2$$
$$= (x^2+1)^2 - x^2$$
$$= (x^2 + 1 - x)(x^2 + 1 + x)$$
$$= (x^2 - x + 1)(x^2 + x + 1)$$

(5)

$$x^4 + 2x^3 + 2x^2 + 2x + 1 = (x^2)^2 + 2x^2 + 1 + 2x^3 + 2x$$
$$= (x^2+1)^2 + 2x(x^2+1)$$
$$= (x^2+1)(x^2 + 1 + 2x)$$
$$= (x^2+1)(x+1)^2$$

(6)

$$x^5 + x + 1 = x^5 - x^2 + x^2 + x + 1$$
$$= x^2(x^3 - 1) + (x^2 + x + 1)$$
$$= x^2(x-1)(x^2+x+1) + (x^2+x+1)$$
$$= (x^2+x+1)(x^2(x-1)+1)$$
$$= (x^2+x+1)(x^3 - x^2 + 1)$$

(7)

$$
\begin{aligned}
(x+y)(x-y)+4(y-1) &= x^2-y^2+4y-4 \\
&= x^2-(y^2-4y+4) \\
&= x^2-(y-2)^2 \\
&= (x-y+2)(x+y-2)
\end{aligned}
$$

(8)

$$
\begin{aligned}
x^3(x-2y)+y^3(2x-y) &= x^4-2x^3y+2xy^3-y^4 \\
&= (x^4-y^4)-2xy(x^2-y^2) \\
&= (x^2-y^2)(x^2+y^2)-2xy(x^2-y^2) \\
&= (x^2-y^2)(x^2+y^2-2xy) \\
&= (x^2-y^2)(x-y)^2 \\
&= (x+y)(x-y)(x-y)^2 \\
&= (x+y)(x-y)^3
\end{aligned}
$$

(9)

$$
\begin{aligned}
& x^2y-y^2z+z^2x-x^2z+y^2x+z^2y-2xyz \\
&= (x^2y-x^2z)+(z^2x+y^2x-2xyz)-(y^2z-z^2y) \\
&= x^2(y-z)+x(z^2+y^2-2yz)-yz(y-z) \\
&= x^2(y-z)+x(y-z)^2-yz(y-z) \\
&= (y-z)(x^2+x(y-z)-yz) \\
&= (y-z)(x^2+xy-xz-yz) \\
&= (y-z)\big(x(x+y)-z(x+y)\big) \\
&= (y-z)(x-z)(x+y)
\end{aligned}
$$

(10)

$1 + a + b + c + ab + bc + ca + abc$
$= (1 + a) + (b + ab) + (c + ca) + (bc + abc)$
$= (1 + a) + b(1 + a) + c(1 + a) + bc(1 + a)$
$= (1 + a)(1 + b + c + bc)$
$= (1 + a)((1 + b) + c(1 + b))$
$= (1 + a)(1 + b)(1 + c)$

(11)

$(ax + by)^2 + (ay - bx)^2 + c^2x^2 + c^2y^2$
$= a^2x^2 + 2abxy + b^2y^2 + a^2y^2 - 2abxy + b^2x^2 + c^2x^2 + c^2y^2$
$= a^2x^2 + b^2x^2 + c^2x^2 + a^2y^2 + b^2y^2 + c^2y^2$
$= x^2(a^2 + b^2 + c^2) + y^2(a^2 + b^2 + c^2)$
$= (a^2 + b^2 + c^2)(x^2 + y^2)$

(12)

$(c + \frac{1}{c} + 1)(c + \frac{1}{c}) = 1$

$\Longrightarrow c^2 + 1 + 1 + \frac{1}{c^2} + c + \frac{1}{c} = 1$

$\Longrightarrow c^2 + c + 1 + \frac{1}{c} + \frac{1}{c^2} = 0$

Multiply by c^2 , where $c \neq 0$

$\Longrightarrow c^4 + c^3 + c^2 + c + 1 = 0$

Multiply by $(c - 1)$, where $c \neq 1$

$\Longrightarrow (c - 1)(c^4 + c^3 + c^2 + c + 1) = 0$
$\Longrightarrow c^5 - 1^5 = 0$
$\Longrightarrow c^5 = 1$
$\Longrightarrow c^{100} = 1$

$\Longrightarrow \left(3c^{100} + \frac{2}{c^{100}} + 1\right)\left(c^{100} + \frac{2}{c^{100}} + 3\right) = (3 + 2 + 1)(1 + 2 + 3) = 36$

Exercises (3-2):

(1)

$(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a)$
$\Longrightarrow (a+b+c)^3 - a^3 - b^3 - c^3 = 3(a+b)(b+c)(c+a)$
$\Longrightarrow (a+b+c)^3 - a^3 - b^3 - c^3 = 3(5)(10)(15)$
$\Longrightarrow (a+b+c)^3 - a^3 - b^3 - c^3 = 2250$

(2)

Assume: $a = x - 2009\, , b = x - 2010\, , c = x - 2011$

$(x-2009)^3 + (x-2010)^3 + (x-2011)^3 = 3(x-2009)(x-2010)(x-2011)$
$\Longrightarrow a^3+b^3+c^3 = 3abc$
$\Longrightarrow a^3+b^3+c^3 - 3abc = 0$
$\Longrightarrow a^3+b^3+c^3 - 3abc = 0$
$\Longrightarrow \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2] = 0$
$\Longrightarrow \frac{1}{2}(a+b+c)[(1)^2+(1)^2+(-2)^2] = 0$
$\Longrightarrow (a+b+c) = 0$
$\Longrightarrow x - 2009 + x - 2010 + x - 2011 = 0$
$\Longrightarrow 3x - 6030 = 0$
$\Longrightarrow x = \frac{6030}{3} = 2010$

(3)

Assume: $x = a+b-c\, , y = c+b-a\, , z = a-b+c$

$\Longrightarrow x+y+z = a+b+c \quad , \quad x+y = 2b \quad , \quad y+z = 2c \quad , \quad x+z = 2a$
$x^3+y^3+z^3 = (x+y+z)^3 - 3(x+y)(y+z)(x+z)$
$\qquad = (a+b+c)^3 - 3(2b)(2c)(2a)$
$\qquad = (a+b+c)^3 - 24abc$

Now we have:

$(a+b+c)^3 - (a+b-c)^3 - (c+b-a)^3 - (a-b+c)^3 - 23abc$
$= (a+b+c)^3 - (x^3+y^3+z^3) - 23abc$
$= (a+b+c)^3 - (a+b+c)^3 + 24abc - 23abc$
$= abc$
$= (2009)(2010)(\frac{1}{2010})$
$= 2009$

(4)

$\sqrt[3]{x-3}+\sqrt[3]{x+3}-\sqrt[3]{x}=0$

Assume: $a=\sqrt[3]{x-3}\ ,\ b=\sqrt[3]{x+3}\ ,\ c=-\sqrt[3]{x}$

$a+b+c=0 \Longrightarrow a^3+b^3+c^3=3abc$

$\Longrightarrow x-3+x+3-x=-3\sqrt[3]{(x-3)(x+3)x}$

$\Longrightarrow x=-3\sqrt[3]{(x^2-9)x}$

$\Longrightarrow x^3=-27(x^2-9)x$

$\Longrightarrow x^3=-27x^3+243x$

$\Longrightarrow 28x^3-243x=0$

$\Longrightarrow x(28x^2-243)=0$

$\Longrightarrow x=0\ \ or\ x=\pm\sqrt{\frac{243}{28}}=\pm\frac{9\sqrt{21}}{14}$

4- Vieta's Formulas:

Exercises:

(1)

$2x^2 - 3x + m = 0$

From **Vieta's formulas**:

$$a + b = \frac{-(-3)}{2} = \frac{3}{2} \quad , \qquad ab = \frac{m}{2}$$

Substitute $a = 2b$

$$a + b = 2b + b = 3b = \frac{3}{2} \Rightarrow b = \frac{1}{2}$$

$$\Rightarrow a = 2b = 2\left(\frac{1}{2}\right) = 1$$

Now substitute the values of a and b into $ab = \frac{m}{2}$

$$\Rightarrow (1)\left(\frac{1}{2}\right) = \frac{m}{2} \Rightarrow \frac{1}{2} = \frac{m}{2} \Rightarrow m = 1$$

(2)

From **Vieta's formulas**:

$a + b + c = 5\,, ab + bc + ac = 12\,, abc = 19$

$i)\ a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ac) = 5^2 - 2(12) = 25 - 24 = 1$

$$ii)\ \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = \frac{c}{abc} + \frac{a}{abc} + \frac{b}{abc} = \frac{a + b + c}{abc} = \frac{5}{19}$$

(3)

The general form of the cubic polynomial whose roots are a, b, cis:

$x^3 - (a + b + c)x^2 + (ab + bc + ac)x - abc = 0$

From the given information, we have:

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = \frac{-3}{32}$$

$$\Rightarrow \frac{a + b + c}{abc} = \frac{-3}{32}$$

$$\Rightarrow a + b + c = \left(\frac{-3}{32}\right)(-64) = 6$$

Also,

$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ac)$

$\Rightarrow 84 = (6)^2 - 2(ab + bc + ac)$

$\Rightarrow 2(ab + bc + ac) = 36 - 84$

$$\Rightarrow ab + bc + ac = \frac{-48}{2} = -24$$

Therefore, the required cubic polynomial is:

$x^3 - 6x^2 - 24x + 64 = 0$

(4)

$$\left(\sqrt[3]{\alpha}\right)^3+\left(\sqrt[3]{\beta}\right)^3+\left(\sqrt[3]{\delta}\right)^3-3(\sqrt[3]{\alpha})(\sqrt[3]{\beta})(\sqrt[3]{\delta})$$
$$=\frac{1}{2}(\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\delta})\left[\left(\sqrt[3]{\alpha}-\sqrt[3]{\beta}\right)^2+\left(\sqrt[3]{\beta}-\sqrt[3]{\delta}\right)^2+\left(\sqrt[3]{\alpha}-\sqrt[3]{\delta}\right)^2\right]$$

$$\Rightarrow \alpha+\beta+\delta-3\sqrt[3]{\alpha\beta\delta}$$
$$=\frac{1}{2}(\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\delta})\left[\left(\sqrt[3]{\alpha}-\sqrt[3]{\beta}\right)^2+\left(\sqrt[3]{\beta}-\sqrt[3]{\delta}\right)^2+\left(\sqrt[3]{\alpha}-\sqrt[3]{\delta}\right)^2\right]$$

$$\Rightarrow -3-3(-1)=\frac{1}{2}(\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\delta})\left[\left(\sqrt[3]{\alpha}-\sqrt[3]{\beta}\right)^2+\left(\sqrt[3]{\beta}-\sqrt[3]{\delta}\right)^2+\left(\sqrt[3]{\alpha}-\sqrt[3]{\delta}\right)^2\right]$$

$$\Rightarrow 0=\frac{1}{2}(\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\delta})\left[\left(\sqrt[3]{\alpha}-\sqrt[3]{\beta}\right)^2+\left(\sqrt[3]{\beta}-\sqrt[3]{\delta}\right)^2+\left(\sqrt[3]{\alpha}-\sqrt[3]{\delta}\right)^2\right]$$

$$\Rightarrow \left(\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\delta}\right)\left[\left(\sqrt[3]{\alpha}-\sqrt[3]{\beta}\right)^2+\left(\sqrt[3]{\beta}-\sqrt[3]{\delta}\right)^2+\left(\sqrt[3]{\alpha}-\sqrt[3]{\delta}\right)^2\right]=0$$

$$\Rightarrow \sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\delta}=0 \;\; or \left[\left(\sqrt[3]{\alpha}-\sqrt[3]{\beta}\right)^2+\left(\sqrt[3]{\beta}-\sqrt[3]{\delta}\right)^2+\left(\sqrt[3]{\alpha}-\sqrt[3]{\delta}\right)^2\right]=0$$

But $\left(\sqrt[3]{\alpha}-\sqrt[3]{\beta}\right)^2+\left(\sqrt[3]{\beta}-\sqrt[3]{\delta}\right)^2+\left(\sqrt[3]{\alpha}-\sqrt[3]{\delta}\right)^2=0$ only if $\sqrt[3]{\alpha}=\sqrt[3]{\beta}=\sqrt[3]{\delta}$

Hence $\alpha=\beta=\delta$. Since $\alpha\beta\delta=-1$, it would follow that $\alpha=\beta=\delta=-1$

and then $\alpha\beta+\beta\delta+\alpha\delta=3$ which contradicts the given value $\alpha\beta+\beta\delta+\alpha\delta=-24$

Therefore, the only possible conclusion is:

$$\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\delta}=0$$

which is exactly what we were required to prove.

Geometry Solutions

exercises 1 Circle:

(1)

a) $\angle ABO = \angle ACO = 90°$

$\Delta ABO \cong \Delta ACO \quad (HL) \Rightarrow$

b) $AB = AC$

c) $\angle AOB = \angle AOC$

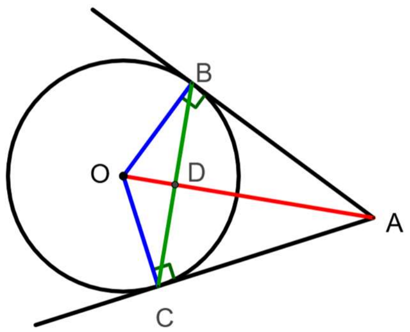
d) $\angle BAO = \angle CAO$

e) $\Delta ABD \cong \Delta ACD \quad (SAS)$

$\Rightarrow m\angle ADB = m\angle ADC$

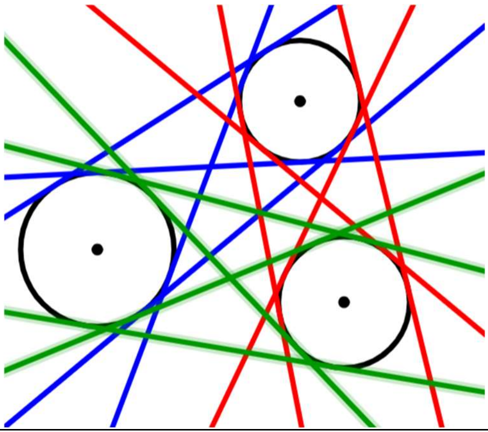
$but\ m\angle ADB + m\angle ADC = 180°$

$\Rightarrow m\angle ADB = m\angle ADC = 90 \ \Rightarrow \overline{AD} \perp \overline{BD}$



(2) Each two circles have 4 common tangents.

Number of common tangents = 12

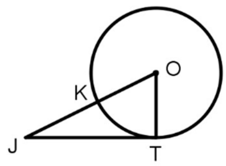


(3)

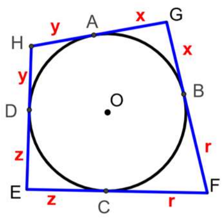
a) $JT = 12$

b) $JT = 10\sqrt{3}$

c) $JT = 15$



(4) $.HG + EF = HE + GF = x + y + z + r$



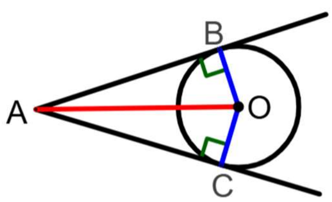
(5)

$\angle ABO = \angle ACO = 90°$

$\Delta ABO \cong \Delta ACO \quad (HL) \Rightarrow$

$\angle BAO = \angle CAO \Rightarrow$

$\angle BAC$ منصف للزاوية $\overrightarrow{AO}$



(6) $\overline{PJ} \perp \overline{JK}$, $\overline{QK} \perp \overline{JK}$

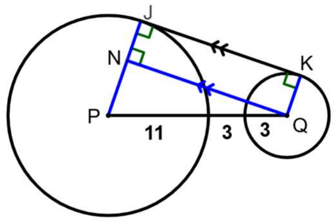
let $\overline{QN} \parallel \overline{JK} \Rightarrow \overline{PJ} \perp QN$

$QK = 3 \Rightarrow JN = 3 \Rightarrow PN = 11 - 3 = 8$

$PQ = 11 + 3 + 3 = 17$

$\Rightarrow QN = \sqrt{17^2 - 8^2} = 15$

$JK = QN = 15$



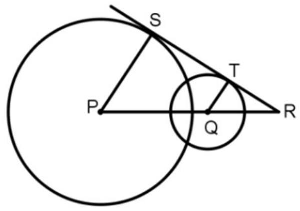
(7) $\overline{QT} \perp \overline{TS}$, $\overline{PS} \perp \overline{TS} \Rightarrow \overline{QT} \parallel \overline{PS}$

$QR = \sqrt{6^2 + 8^2} = 10 \Rightarrow QP = 30 - 10 = 20$

$\Delta QTR \sim \Delta PSR$ (AA)

$\frac{QT}{PS} = \frac{QR}{PR} \Rightarrow \frac{6}{PS} = \frac{10}{30} \Rightarrow PS = 18$

$\frac{RT}{TS} = \frac{RQ}{QP} \Rightarrow \frac{8}{TS} = \frac{10}{20} \Rightarrow TS = 16$



(8) $\overline{PJ} \perp \overline{JK}$, $\overline{QK} \perp \overline{JK}$

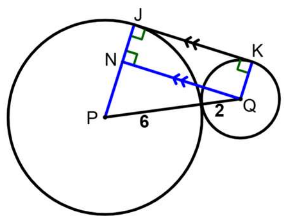
let $\overline{QN} \parallel \overline{JK} \Rightarrow \overline{PJ} \perp QN$

$QK = 2 \Rightarrow JN = 2 \Rightarrow PN = 6 - 2 = 4$

$PQ = 6 + 2 = 8$

$\Rightarrow QN = \sqrt{8^2 - 4^2} = 4\sqrt{3}$

$JK = QN = 4\sqrt{3}$

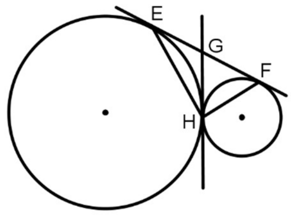


(9) $GF = GH \Rightarrow \angle GFH = \angle GHF = x$

$GE = GH \Rightarrow \angle GEH = \angle GHE = y$

$\Delta EHF: 2x + 2y = 180 \Rightarrow x + y = 90$

$\angle EHF = x + y = 90$

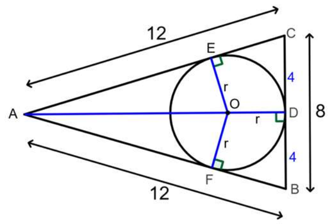


(10) $\Delta ADC: AD = \sqrt{12^2 - 4^2} = 8\sqrt{2}$

$OD = r \Rightarrow AO = 8\sqrt{2} - r$

$\Delta ADC \sim \Delta AEO$ $(AA) \Rightarrow \frac{AO}{AC} = \frac{OE}{CD}$

$\Rightarrow \frac{8\sqrt{2} - r}{12} = \frac{r}{4} \Rightarrow r = 2\sqrt{2}$

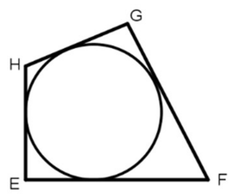


(11)

$HG + EF = HE + GF$

$\Rightarrow 12 + 16 = HE + 15$

$\Rightarrow HE = 13$



exercises 2 Arcs and Central angles:

(1)	$m\angle 1 = 85°$	(2)	$\boldsymbol{m\angle 1 = 80°}$
(3)	$m\angle 1 = 150°$	(4)	$\boldsymbol{m\angle 1 = 50°}$
(5)	$m\angle 1 = 120°$	(6)	$\boldsymbol{m\angle 1 = 55°}$

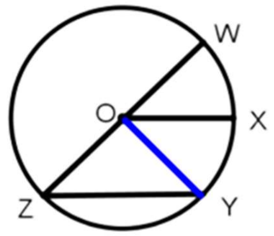
(7) $\overline{OX} \parallel \overline{ZY}$

$\Rightarrow \angle WOX = \angle OZY \,, \angle OYZ = \angle XOY$

$OY = OZ \Rightarrow \angle OZY = \angle OYZ$

$\therefore \angle WOX = \angle OZY = \angle OYZ = \angle XOY$

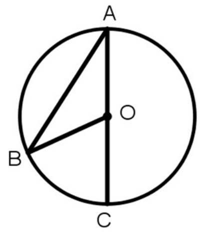
$\Rightarrow \angle WOX = \angle XOY \Rightarrow \widehat{WX} \cong \widehat{XY}$



(8) (a $m\angle B = 35°$, $m\angle BOC = 70°$, $m\,\widehat{BC} = 70°$

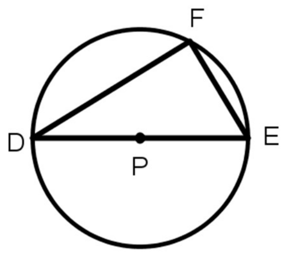
(b $m\,\widehat{BC} = 2n$

(c $m\angle A = 3k$



(9) $\angle DEF = 90 - \frac{n}{2}$ (a

$\angle DFE = 90$ (b



Exercises 3 Arcs and chords:

(1)

a) $\overline{AB} \cong \overline{CD}$, $OA = OD = OB = OC = r$

$\Rightarrow \Delta OAB \cong \Delta OCD (SSS)$

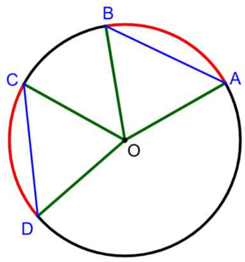
$\Rightarrow \angle AOB \cong \angle COD \;\Rightarrow \widehat{AB} \cong \widehat{CD}$

b) $\widehat{AB} \cong \widehat{CD} \Rightarrow \angle AOB \cong \angle COD$

$OA = OD = OB = OC = r$

$\Rightarrow \Delta OAB \cong \Delta OCD (SAS)$

$\Rightarrow \overline{AB} \cong \overline{CD}$



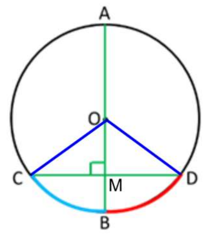
(2)

$OC = OD = r$, $OM = OM$, $\angle OMD = \angle OMC = 90°$

$\Rightarrow \Delta OMD \cong \Delta OMC (HL)$

$\Rightarrow MD = MC$, $\angle MOD = \angle MOC$

$\Rightarrow \widehat{BC} \cong \widehat{BD}$



(3)

a) $AB = CD \Rightarrow AF = CE$, $OA = OC = r$,

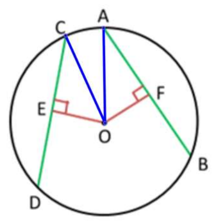
$\Rightarrow \Delta OFA \cong \Delta OEC (HL)$

$\Rightarrow OF = OE$

b) $OF = OE$, $OA = OC = r$,

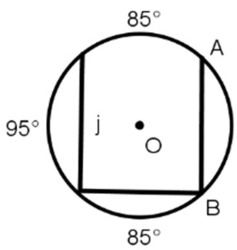
$\Rightarrow \Delta OFA \cong \Delta OEC (HL)$

$\Rightarrow AF = CE \;\Rightarrow AB = CD$



(4)

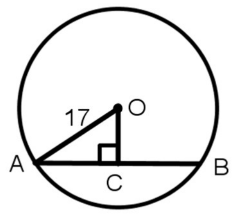
$\overline{AB} = j$



(5)

$OC = \sqrt{17^2 - 15^2}$

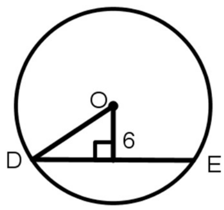
$OC = 8$



(6)

$OD = \sqrt{8^2 + 6^2}$

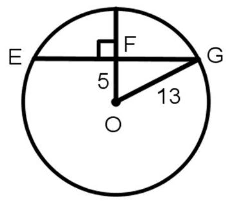
$OD = 10$



(7)

$FG = \sqrt{13^2 - 5^2}$

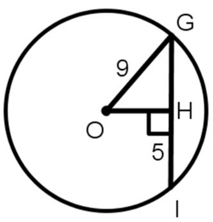
$EG = 20$



(8)

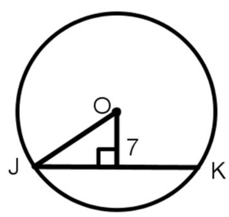
$OH = \sqrt{9^2 - 5^2}$

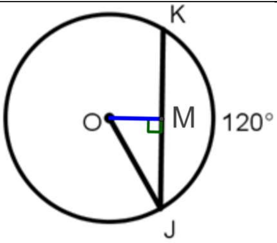
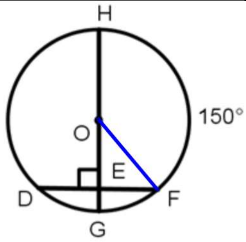
$OH = 2\sqrt{14}$



(9)

$OJ = 7\sqrt{2}$



(10) $\angle JOM = \frac{120}{2} = 60°$ $\Rightarrow JM = 6\sqrt{3}$ $\Rightarrow JK = 12\sqrt{3}$ 	(11) $\overline{OE} = 8\sqrt{3}$ $\angle EOF = 180 - 150$ $= 30°$ $\Rightarrow \text{OF} = 16 = OH$ $HG = 2 \times 16 = 32$ 

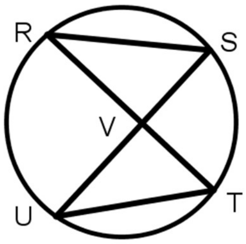
(12) $\overline{RS} \cong \overline{UT}$

$,\angle R \cong \angle U$,

$\angle RVS \cong \angle UVT$

$\Rightarrow \Delta RVS \cong \Delta UVT \quad (AAS)$

$\Rightarrow \overline{VS} \cong \overline{VT}\ , \overline{RV} \cong \overline{UV}$

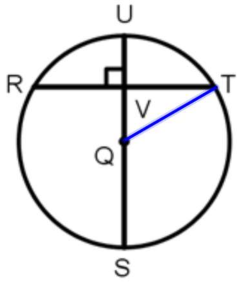


(13) $QT = QS = 10$

$VT = \frac{16}{2} = 8$

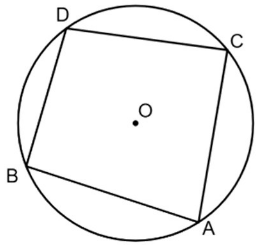
$\Rightarrow QV = \sqrt{10^2 - 8^2} = 6$

$\Rightarrow UV = 10 - 6 = 4$

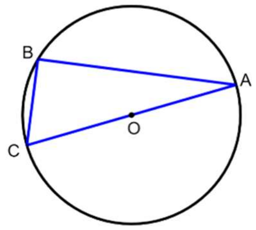


Exercises 4 peripheral and tangential angles:

(1) $m\angle A + m\angle D = \frac{1}{2}(m\,\widehat{BDC} + m\,\widehat{BAC}) = \frac{1}{2}(360°) = 180°$



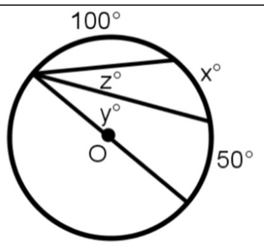
(2) $m\angle B = \frac{1}{2}(m\,\widehat{CA}) = (180°) = 90°$



(3) $x = 180 - (100 + 50)$
$= 30°$

$y = \frac{50}{2} = 25°$

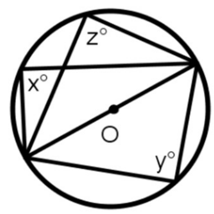
$z = \frac{30}{2} = 15°$



(4) $x = 90°$

$y = 90°$

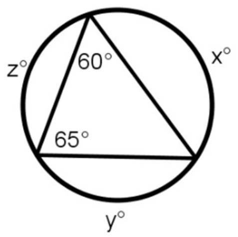
$z = 90°$



(5) $x = 2 \times 65 = 130°$

$y = 2 \times 60 = 120°$

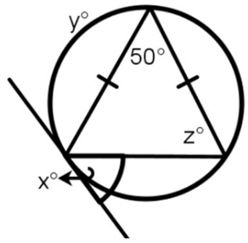
$z = 360 - (130$
$+ 120)$
$= 110°$



(6) $x = 50°$

$z = \frac{1}{2}(180 - 50) = 65$

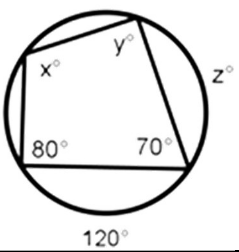
$y = 2 \times 65 = 130°$



(7) $x = 180 - 70 = 110°$

$y = 180 - 80 = 100°$

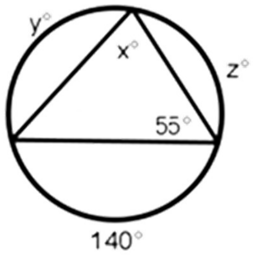
$z = 2 \times 110 - 120$
$= 100°$



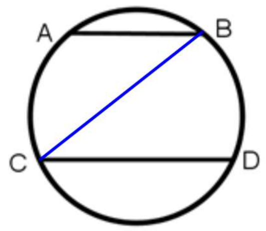
(8) $x = \frac{140}{2} = 70°$

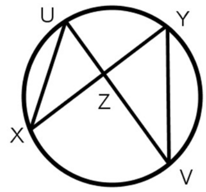
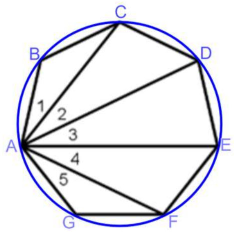
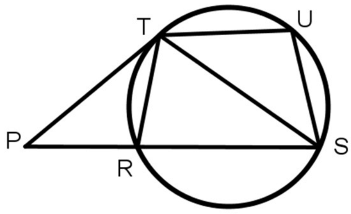
$y = 2 \times 55 = 110°$

$z = 360 - (140$
$+ 110)$
$= 110°$



(9) $\overline{AB} \parallel \overline{CD} \Rightarrow \angle ABC = \angle BCD$
$\Rightarrow \widehat{AC} \cong \widehat{BD}$



(10)	$\angle XUV \cong \angle VYX$ $\angle YVU \cong \angle YXU$ $\Rightarrow \Delta UXZ \sim \Delta YVZ \ (AA)$	
(11)	$\angle BAG = \frac{(7-2)\times 180}{7} = \frac{900°}{7}$ $\overline{BC} \cong \overline{CD} \cong \overline{DE} \cong \overline{EF} \cong \overline{FG}$ $\Rightarrow \widehat{BC} \cong \widehat{CD} \cong \widehat{DE} \cong \widehat{EF} \cong \widehat{FG}$ $\Rightarrow \angle 1 \cong \angle 2 \cong \angle 3 \cong \angle 4 \cong \angle 5 = \frac{180°}{7}$	
(12)	$\overline{TU} \parallel \overline{PS}$ $\angle UTS \cong \angle RST \cong \angle RTP$ $\angle TUS \cong \angle STP \cong \angle TRP$ $\Rightarrow \Delta TUS \sim \Delta STP \sim \Delta TRP \ \ (AA)$	

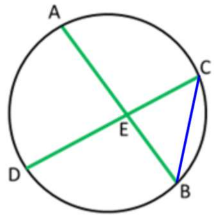
Exercises 5 Angles Between Two Secant Lines:

(1)

$\angle \boldsymbol{BCD} = \frac{1}{2}\widehat{BD}$

$\angle \boldsymbol{ABC} = \frac{1}{2}\widehat{AC}$

$\angle AEC = \angle C + \angle B = \frac{1}{2}[\widehat{AC} + \widehat{BD}]$

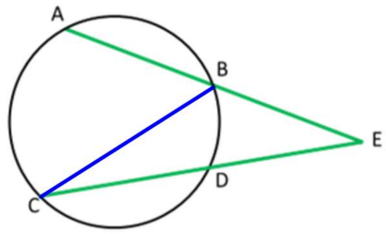


(2)

$\angle \boldsymbol{BCD} = \frac{1}{2}\widehat{BD}$

$\angle \boldsymbol{ABC} = \frac{1}{2}\widehat{AC}$

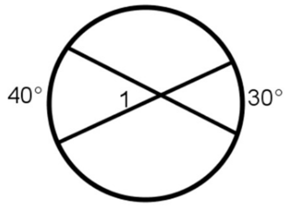
$\angle AEC = \angle ABC - \angle BCD = \frac{1}{2}[\widehat{AC} - \widehat{BD}]$



(3)

$m\angle 1 = \frac{30+40}{2}$

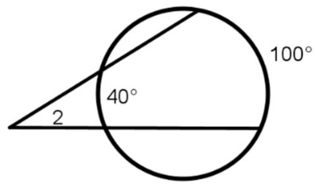
$= 35°$



(4)

$m\angle 2 = \frac{100-40}{2}$

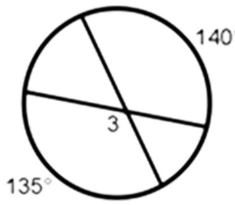
$= 30°$



(5)

$m\angle 3 = \frac{140+135}{2}$

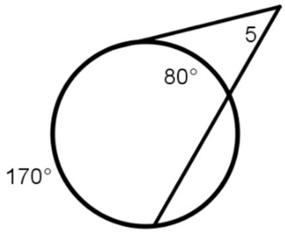
$= 137.5°$



(6)

$m\angle 5 = \frac{170-80}{2}$

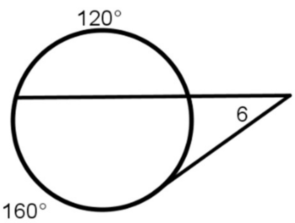
$= 45°$



(7)

$m\angle 6 = \frac{160-80}{2}$

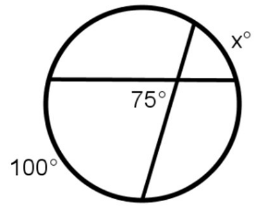
$= 40°$



(8)

$75 = \frac{100+x}{2}$

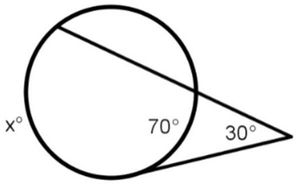
$\Rightarrow x = 50°$



(9)

$30 = \frac{x-70}{2}$

$\Rightarrow x = 130°$

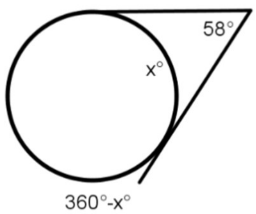


(10)

$58 = \frac{360-x-x}{2}$

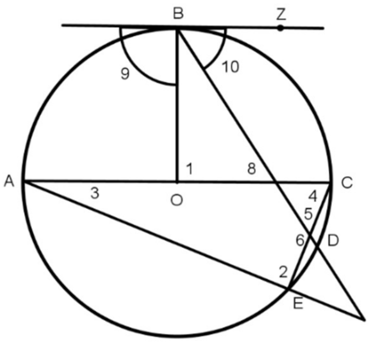
$\Rightarrow 58 = 180 - x$

$\Rightarrow x = 122°$



(11)

$$m\angle 1 = \overset{\frown}{BC} = 90°$$
$$m\angle 2 = \frac{1}{2}\overset{\frown}{ABC} = 90°$$
$$m\angle 3 = \frac{1}{2}\overset{\frown}{CE} = 25°$$
$$m\angle 4 = \frac{1}{2}\overset{\frown}{AE} = 65°$$
$$m\angle 5 = \frac{\overset{\frown}{BC} + \overset{\frown}{DE}}{2} = 55°$$
$$m\angle 6 = 180 - m\angle 5 = 125°$$
$$m\angle 8 = \frac{\overset{\frown}{AB} + \overset{\frown}{CD}}{2} = 60°$$
$$m\angle 9 = 90°$$
$$m\angle 10 = \frac{1}{2}\overset{\frown}{BD} = 60°$$



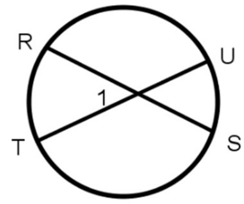
(12)

a) $m\angle 1 = \frac{80+40}{2} = 60°$

b) $m\angle 1 = \frac{130+100}{2} = 115°$

c) $m\,\overset{\frown}{US} = 30°$

d) $m\,\overset{\frown}{RT} = 68°$

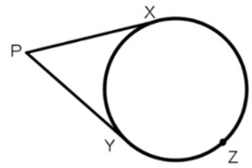


(13)

a) $m\angle P = 250 - 180 = 70°$

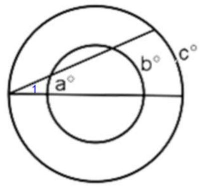
b) $m\angle P = 180 - 90 = 90°$

c) $m\,\overset{\frown}{XY} = 180 - 85 = 95°$



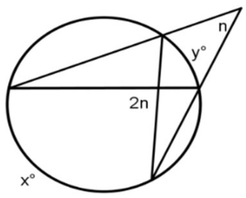
(14)

$$m\angle 1 = \frac{1}{2}C$$
$$m\angle 1 = \frac{b-a}{2}$$
$$\Rightarrow c = b - a$$



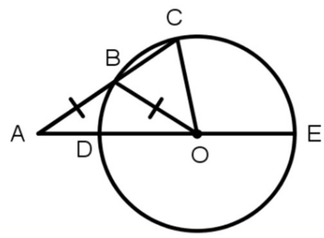
(15)

$$\frac{x+y}{2} = 2n \Rightarrow x + y = 4n$$
$$\frac{x-y}{2} = n \Rightarrow x - y = 2n$$
$$\Rightarrow x + y = 2x - 2y \Rightarrow x = 3y$$
$$\Rightarrow x : y = 3 : 1$$



(16)

$$\angle A = \frac{\overset{\frown}{CE} - \overset{\frown}{BD}}{2}$$
$$\angle BOD = \overset{\frown}{BD}$$
$$AB \cong BO \Rightarrow \angle A \cong \angle BOD$$
$$\Rightarrow \frac{\overset{\frown}{CE} - \overset{\frown}{BD}}{2} = \overset{\frown}{BD} \Rightarrow \overset{\frown}{CE} = 3\,\overset{\frown}{BD}$$



Solutions for Remainders Exercises:

(1)

The remainder of 5 when divided by 7 is 5 since:

$$5 = 0 \times 7 + 5$$

(2)

Similar to the previous question, the remainder is m since:

$$m = 0 \times n + m$$

(3)

a) Note in this question, we do not need to divide the entire number $2015 \cdot 2016 \cdot 2017 + 2018^3$ by 9. It is sufficient to replace each of the numbers with the remainder of their division by 9 (from the remainder theorem). Thus, the result becomes the remainder of dividing 8 × 0 × 1 + 2^3 by 9. Consequently, the remainder is 8.

b) Similar to the idea in the previous question the remainder of 8^{100} when divided by 7 is similar to the remainder of 1^{100} when divided by 7. Thus, the answer is 1.

(4)

Since the remainder of N_1 by 3 is r_1, we can write the number N_1 as follows:

$$N_1 = k_1 \times 3 + r_1.$$

Similarly, we have:

$$N_2 = k_2 \times 3 + r_2.$$

Therefore:

$$N_1 + N_2 = 3 \times (k_1 + k_2) + r_1 + r_2.$$

Notice that $3 \times (k_1 + k_2)$ is divisible by 3. Thus, the remainder of $N_1 + N_2$ by 3 is $r_1 + r_2$.

(5)

In exercises 5-8, when we try many numbers, all statements seem to be correct. But how do we prove their validity for all numbers n?

The idea in these problems is to replace n with the possible remainders when dividing it by the number mentioned in the question, and then ensure that the possible remainders of the results satisfy the requirement.

The possible remainders for n when divided by 5 are $\{0, 1, 2, 3, 4\}$. Therefore, the possible remainders for $n^4 + 4n$ are:

$$0^5 + 4 \cdot 0 = 0,$$
$$1^5 + 4 \cdot 1 = 5,$$
$$2^5 + 4 \cdot 2 = 40,$$
$$3^5 + 4 \cdot 3 = 255,$$
$$4^5 + 4 \cdot 4 = 1040.$$

Since all possible remainders are divisible by 5, then the number $n^4 + 4n$ must be divisible by 5.

(6)

The possible remainders for n when divided by 3 are $\{0, 1, 2\}$. Therefore, the possible remainders for $n^2 + 1$ are:

$$0^2 + 1 = 1,$$
$$1^1 + 1 = 2,$$
$$2^2 + 1 = 5,$$

Since all possible remainders are NOT divisible by 3, then the number $n^2 + 1$ cannot be divisible by 3.

(7)

The possible remainders for n when divided by 3 are $\{0,1,2,3,4,5,6,7,8\}$. Therefore, the possible remainders for n^3 are:

$$n^3 = \{0^3, 1^3, 2^3, 3^3, 4^3, 5^3, 6^3, 7^3, 8^3\} = \{0,1,8,0,1,8,0,1,8\} = \{0,1,8\}$$

And the possible remainders for $n^3 + 2$ are:

$$n^3 + 2 = \{0 + 2, 1 + 2, 8 + 2\} = \{2,3,1\}$$

Since all possible remainders are NOT divisible by 9, then the number $n^3 + 2$ cannot be divisible by 9.

(8)

In this question, we must prove that the number $n^3 - n$ is divisible by 3 and 8 in order to prove that it is divisible by 24.

The possible remainders for n when divided by 3 are $\{0, 1, 2\}$. Thus, the possible remainders for $n^3 - n$ are:

$$0^3 - 0 = 0,$$
$$1^3 - 1 = 0,$$
$$2^3 - 2 = 6.$$

Since all possible remainders are divisible by 3, the number $n^3 - n$ must be divisible by 3.

Similarly, the possible remainders for n when divided by 8 are $\{1, 3, 5, 7\}$ because if the remainder were even, n would be even, which is impossible. Thus, the possible remainders for n^3 - n are:

$$1^3 - 1 = 0,$$
$$3^3 - 3 = 24,$$
$$5^3 - 5 = 120,$$
$$7^3 - 7 = 336.$$

Since all possible remainders are divisible by 8, the number $n^3 - n$ must be divisible by 8. Since it is divisible by 3 and 8, it is therefore divisible by 24.

(9)

In this question and the one following it, we notice the following property of prime numbers greater than 3. That is, any prime number $p > 3$ can be written in one of two forms:

$$p = 6k + 1 \text{ or } p = 6k - 1.$$

This is because the possible remainders of p cannot be $\{0, 2, 3, 4\}$ because the number would not be prime in that case (for example: if the remainder of p is 4, then $p = 6k + 4 = 2(3k + 2)$, and this is not prime because it is divisible by 2 and $3k + 2$).

Therefore, when substituting the possible forms for p, we notice that:

$$p^2 - 1 = (6k \pm 1)^2 - 1 = 36k^2 \pm 12k + 1 - 1 = 12(3k^2 \pm k).$$

We notice that $3k^2 \pm k$ is an even number. Thus, $p^2 - 1$ is divisible by 24 because it is 12 multiplied by an even number.

(10)

Note that any prime number $p > 3$ can be written in one of two forms:

$$p = 6k_1 + 1 \text{ } or \text{ } p = 6k_1 - 1.$$

Similarly:

$$q = 6k_2 + 1 \text{ } or \text{ } q = 6k_2 - 1.$$

Consequently:

$$p^2 - q^2 = (6k_1 \pm 1)^2 - (6k_2 \pm 1)^2 = 36k_1^2 \pm 12k_1 + 1 - 36k_2^2 \pm 12k_2 - 1$$
$$= 36(k_1^2 - k_2^2) + 12(\pm k_1 \pm k_2) = 12(3k_1^2 \pm k_1 + 3k_2^2 \pm k_2).$$

But (3k_1^2 ± k_1 + 3k_2^2 ± k_2)

is an even number. Thus, p^2 - q^2 is divisible by 24.

Another way to solve this is to prove that $p^2 - q^2$ is divisible by 3 and 8.

Since p is a prime number greater than 3, notice that the possible remainders for p when divided by 3 are $\{1, 2\}$ and when divided by 8 are $\{1, 3, 5, 7\}$.

Therefore, the possible remainders for p^2 when divided by 3 are:

$$p^2 = \{1^2, 2^2\} = \{1\},$$

and when divided by 8: $p^2 = \{1^2, 3^2, 5^2, 7^2\} = \{1\}$.

The same applies to q. Thus, the possible remainders for $p^2 - q^2$ when divided by 3 or 8 are:

*p*2– *q*2= 1 – 1= 0.

Consequently, the number p^2 - q^2 is divisible by 24.

(11)

Since we have an equality between two values, both values must have the same remainder when divided by any number.

We will try 3 (because the question asked to prove divisibility by 3).

We assume, for the sake of reaching a contradiction, that none of x, y, or z is divisible by 3. Therefore, the possible remainders for x^2, y^2, and z^2 when divided by 3 are:

$$x^2 = \{1^2, 2^2\} = \{1\},$$

and similarly for y^2 and z^2.

However, we have the equation:

$$x^2 + y^2 = z^2.$$

The remainder of dividing the left side by 3 is 2, while the remainder of dividing the right side by 3 is 1. This contradicts the assumption.

Consequently, at least one of the numbers x, y, or z must be divisible by 3.

(12)

We will prove that the two numbers a and b are both divisible by 3 and 7.

Consequently, a^2 and b^2 are both divisible by 9 and 49, and therefore:

$$441 = 9 \times 49 \mid a^2 + b^2.$$

Note that the possible remainders for a and b when divided by 3 are $\{0, 1, 2\}$.

Therefore, the possible remainders for a^2 and b^2 are:

$$a^2, b^2 = \{0^2, 1^2, 2^2\} = \{0, 1\}.$$

Thus, the possible remainders for $a^2 + b^2$ are:

$$a^2 + b^2 = \begin{cases} 0 + 0 = 0 \\ 0 + 1, 1 + 0 = 1 \\ 1 + 1 = 2 \end{cases}$$

However, it is given that $a^2 + b^2$ is divisible by 3. Therefore, the remainder for $a^2 + b^2$ must equal zero.

This is not achieved unless both a and b are divisible by 3.

Similarly, the possible remainders for a and b when divided by 7 are $\{0, 1, 2, 3, 4, 5, 6\}$. Consequently, the possible remainders for a^2 and b^2 are:

$$a^2, b^2 = \{0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2\} = \{0, 1, 4\}.$$

Thus, the possible remainders for $a^2 + b^2$ are:

$$a^2 + b^2 = \begin{cases} 0 + 0 = 0 \\ 0 + 1, = 1 \\ 0 + 4 = 4 \\ 1 + 1 = 2 \\ 1 + 4 = 5 \\ 4 + 4 = 8 \end{cases}$$

However, it is given that $a^2 + b^2$is divisible by 7. Therefore, the remainder for $a^2 + b^2$ must equal zero. This is not achieved unless both a and b are divisible by 7. Since both a and b are divisible by 3 and 7, then:

$$3^2 \mid a^2 + b^2, 7^2 \mid a^2 + b^2 \Rightarrow 441 \mid a^2 + b^2.$$

(13)

Note that the remainder of a divided by 6 is the same as the remainder of a^3 divided by 6 because:

$$a = \{0,1,2,3,4,5\} \text{ and } a^3 = \{0^3,1^3,2^3,3^3,4^3,5^3\} = \{0,1,8,27,64,125\} = \{0,1,2,3,4,5\}.$$

Consequently, the remainder of $a^3 + b^3 + c^3$ is equal to the remainder of $a + b + c$, which is divisible by 6.

(14)

We will assume, for the sake of reaching a contradiction, that d is not divisible by 6.

Consequently, the remainder of dividing d by 6 is $\{1,2,3,4,5\}$. Since p, q, and r are prime numbers greater than 3, p can be written in the form:

$$p = 6k + 1 \text{ or } p = 6k - 1.$$

We will take both cases with the possible remainders for d and prove that either q or r will not be prime.

Case 1: $p = 6k + 1$, then the remainder of q, r will be:

$$q = p + d = \begin{cases} 6k+1+1 = 2(3k+1) \\ 6k+1+2 = 3(2k+1) \\ 6k+1+3 = 2(3k+2) \\ 6k+1+4 = 6k+5 \\ 6k+1+5 = 6(k+1) \end{cases}, r = p + 2d = \begin{cases} 6k+1+2 = 3(2k+1) \\ 6k+1+4 = 6k+5 \\ 6k+1+6 = 6k+7 \\ 6k+1+8 = 3(2k+3) \\ 6k+1+10 = 6k+11 \end{cases}$$

Notice in all of the cases, one of q, r is not a prime, which leads to a contradiction.

Case 2: is similar to case 1 by replacing $p = 6k - 1$ we will reach the same contradiction.

Mixed Exercises

(1)

We choose one person for the first room, then two people for the second room, and finally the remaining students for the last room.
(Note that the last combination can be ignored, since the remaining four students will necessarily go to the last room.) $\binom{7}{1} \times \binom{6}{2} \times \binom{4}{4} = 105$

(2)

All 10-digit numbers: 9×10^9

Those with all digits different: $9 \times 9 \times 8 \times 7 \times \ldots \times 1$

Therefore, the required number is: $9 \times 10^9 - 9 \times 9!$

(3)

We consider the possible cases for the number of B's and add them together:

$$\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \binom{8}{3}$$

(4)

The total number is: 9×10^6

Those that do not contain the digit "1" $= 8 \times 9^6$

This is not equal to half, therefore the answer is: No.

(5)

We look at the first 9 digits from the left.
If their sum is even (call it AAA), then the units digit must be even and has 5 choices. If their sum is odd (call it BBB), then the units digit must be odd and has 5 choices.

So the total is: $5A + 5B = 5(A + B)$

And $A + B$ is the number of 9-digit numbers, which equals 9×10^8

Final answer: $5 \times 9 \times 10^8$

(6)

We choose a row and a column for each rook. No row or column is repeated, since two rooks cannot be in the same row or column.

Then we divide by $4!$ because the rooks are identical: $\frac{(8\times8)\times(7\times7)\times(6\times6)\times(5\times5)}{4!}$

(7)

We treat A and B as one block together with the remaining four books, so there are *5*! arrangements. Inside the block, there are two possible orders (AB or BA):

$$5! \times 2$$

(8)

Since Jawad is fixed in the committee, we chose two people from the remaining nine:

$$\binom{9}{2}$$

Ascending and Descending Order

(1)

Note that choosing 3 digits is enough to determine whether the order is ascending or descending. In the ascending case, zero cannot be chosen because it would occupy the hundreds place.

Ascending: choose 3 different digits from 1,2..,9 : $\binom{9}{3}$

Descending: choose 3 different digits from 0,1..,9 : $\binom{10}{3}$

So the total number is: $\binom{9}{3} + \binom{10}{3} = 204$

(2)

Same idea as the previous question, so the answers for the two parts are equal:

$$\binom{8}{5} = \binom{8}{3} = 56$$

(3)

In the ascending case, the solution is different because zero cannot be chosen:

(a) $\binom{8}{4}$

(b) $\binom{9}{4}$

(4)

The number of ways to draw 3 cards with order is: $10 \times 9 \times 8$

Salem wins (ascending order): $\binom{10}{3} = 120$

Since each set of 3 cards has only one ascending arrangement, Mohammed has a higher chance of winning.

Result: Mohammed wins in about 16.7% of the cases.

Counting Paths on a Grid

(1)

We need 6 forward jumps and 7 backward jumps to reach −1. The number of ways is the number of ways to choose 6 forward jumps out of 13 jumps:

$\binom{13}{6}$ Or $\binom{13}{7}$ and they are equal.

(2)

48

(3)

Each rectangle is determined by choosing two vertical lines and two horizontal lines: $\binom{10}{2} \times \binom{8}{2} = 1260$

(4)

We have 25 letters. We choose two of them using combinations (and they have only one order since they are alphabetically ordered). We also choose two digits from 9 digits:

$$\binom{25}{2} \times \binom{9}{2} = 10800$$

(5)

There are 3 vowels (A, O, U). We choose 2 of them: $\binom{3}{2} = 3$

There are 6 consonants: M, T, T, H, C, N.

- 3 consonants with no T: $\binom{4}{3} = 4$
- 3 consonants with one T: $\binom{4}{2} = 6$
- 3 consonants with two T's: $\binom{4}{1} = 4$

Since in all cases there are 4 non-T letters, the total is: $3(4+6+4) = 42$

Stars and Bars

(1)

We place one box in each room initially to satisfy the condition that each room has at least one box. This leaves 7 boxes to be distributed among the three rooms. Using stars and bars: $\binom{7+3-1}{3-1} = \binom{9}{2} = 36$

(2)

We park 5 cars in a row of parking spaces. There are 5! ways to arrange the cars, and we must ensure at least one empty space between each pair of cars. After placing the 4 gaps, one empty space remains, which can be placed in any of the 6 possible gaps. (before the first car, between any two cars, or after the last car). Thus, the total number of arrangements is: = $6 \cdot 5!$

(3)

Applying stars and bars to distribute 80 items into 4 groups:

$$\binom{80+4-1}{4-1} = \binom{83}{3}$$

(4)

Since all solutions must be positive, we subtract 1 from each variable, so that we are left with 76 non-negative units. Now we apply the stars and bars method:

$$\binom{76+4-1}{4-1} = \binom{79}{3}$$

(5)

The number of ways to choose 4 empty parking spots in general can be found directly by combinations:

$$\binom{16}{4} = 1820$$

Now we calculate the complement, which is the distribution of the spots under the condition that no two spots are adjacent. We place the 12 cars in a row. We have 4 unused spots that can be placed in the 13 gaps between the cars and at the ends.

We choose 4 positions out of 13 to place the empty spots:

$$\binom{13}{4} = 715$$

Thus, there are:

$$1820 - 715 = 1105$$

ways in which Sajad can park with probability:

$$\frac{1105}{1820} = 60\%$$

(6)

To calculate the number of ways to seat 3 teachers and 6 students on 9 chairs such that each teacher sits between two students, we exclude the two ends, so teachers cannot sit there. This leaves 7 middle seats. We choose 3 seats for the teachers, leaving one empty chair between every two teachers to ensure that they are not adjacent. The number of ways to arrange the teachers is 3!. Then we distribute the remaining two chairs among 4 available positions (before the first teacher, between the teachers, and after the third teacher) using the stars and bars method. $\begin{pmatrix} 2+4-1 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10$

The final result is then multiplied by the number of ways to arrange the students:

$$3! \cdot 10 \cdot 6! \ = \ 43200$$

Challenge Questions

(1)

Neither of them is chosen: $\binom{10}{4}$

Or only one of them is chosen: $2.\binom{10}{3}$

Total: $2.\binom{10}{3}+\binom{10}{4}$

(2)

Since the even number must have the smallest digit even (because it is descending), we must choose 6 digits from 0 to 9 such that the smallest one is even.

We split it into cases: either 0 is the smallest and we choose 5 digits from 1 to 9, or 2 is the smallest and we choose 5 digits from 3 to 9, or 4 is the smallest and we choose 5 digits from 5 to 9. Note that 6 or 8 cannot be the smallest because there are not enough digits.

Answer: $\binom{9}{5}+\binom{7}{5}+\binom{5}{5}=148$

(3)

First, we give each child one piece of candy to satisfy the first condition.

This leaves 12 pieces, which we distribute among 3 children, such that the first child takes at most 6 pieces. The number of ways without the restriction, using the stars and bars method, is: $\binom{12+3-1}{3-1}=\binom{14}{2}=91$

Now we give the first child 7 pieces out of the 12 in order to count the number of ways that do not satisfy the condition.

This leaves 5 pieces to be distributed among 3 children: $\binom{5+3-1}{3-1}=\binom{7}{2}=21$

And the final answer is: $91-21=70$

(4)

We choose a different row and a different column for each rook so that they do not attack each other. Then we divide by 4!4!4! because the first four rooks are identical in color, and we also divide by another 4!4!4! because the second four rooks are identical in color.

The answer is:

$$\frac{(8\times 8)\times(7\times 7)\times\ldots\times(1\times 1)}{4!\times 4!}$$

(5)

There are 50! ways to arrange the teachers at the tables, then we arrange the students as well in 50! ways since the tables are distinct. Now, at each table there are two possible orders for the student and the teacher, and since the seats are distinct, we must multiply by 2^{50}

And the answer is: $50! \times 50! \times 2^{50}$

(6)

We consider the two adjacent students as one unit. Thus, we have 7 elements to arrange around a table in $6!$ *ways (circular arrangement). Then we multiply by 2 to account for swapping their positions.*

Now we subtract the cases in which the third person is adjacent to the first two students; either to their right or to their left, and arrange the remaining five in $5!$ *ways. Then we multiply by 2 to swap their positions.*

Thus, the final answer is:

$$2 \times 6! - 2 \times 2 \times 5! = 2 \times 480 = 960$$

نسمو
NSMO
National Science and Mathematics Olympiad

وزارة التعليم
Ministry of Education

موهبة
Mawhiba